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Module 2: Series expansion and Multivariable Calculus

Taylor's and Maclaurin's Series Expansion

① Taylor's series expansion of $f(x)$ about the point $x=a$ is given by

$$y(x) = y(a) + (x-a)y_1(a) + \frac{(x-a)^2}{2!}y_2(a) + \frac{(x-a)^3}{3!}y_3(a) \dots$$

② Maclaurin's Series expansion of $f(x)$ about the point $x=a=0$ is given by

$$y(x) = y(0) + x^1 \cdot y_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) \dots\dots$$

Problems.

① Expand $\sin x$ about the point $x=\pi/2$ upto 4^{th} degree term using Taylor's Series Expansion.

Taylor's series exp. is given by

$$y(x) = y(a) + (x-a)y_1(a) + \frac{(x-a)^2}{2!}y_2(a) + \frac{(x-a)^3}{3!}y_3(a) + \frac{(x-a)^4}{4!}y_4(a)$$

Here $a = \frac{\pi}{2}$

$$y(x) = y(\pi/2) + (x-\pi/2)y_1(\pi/2) + \frac{(x-\pi/2)^2}{2!}y_2(\pi/2) + \frac{(x-\pi/2)^3}{3!}y_3(\pi/2)$$

$$+ \frac{(x-\pi/2)^4}{4!}y_4(\pi/2) \rightarrow ①$$

Let $y(x) = \sin x$; $y(a) = y(\pi/2) = \sin \pi/2 = 1$

$y_1 = \cos x$; $y_1(\pi/2) = \cos \pi/2 = 0$

$y_2 = -\sin x$; $y_2(\pi/2) = -\sin(\pi/2) = -1$

$y_3 = -\cos x$; $y_3(\pi/2) = 0$

$y_4 = +\sin x$; $y_4(\pi/2) = 1$

Sub in ①

$$y(x) = 1 + (x-\pi/2)\times 0 + \frac{(x-\pi/2)^2}{2!}(-1) + 0 + \frac{(x-\pi/2)^4}{4!}\cdot 1$$

$$y(x) = 1 - \frac{(x-\pi/2)^2}{2} + \frac{(x-\pi/2)^4}{24}$$

② Obtain the Maclaurin's series expansion of $\log(1+x)$ upto the term containing x^4

Maclaurin's series exp is given by

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0) \rightarrow ①$$

Consider $y(x) = \log(1+x)$; $y(0) = \log(1+0) = \log 1 = \boxed{0}$

$y_1(x) = \frac{1}{1+x}\times(0+1) = \frac{1}{1+x}$; $y_1(0) = \frac{1}{1+0} = \boxed{1}$

$y_2(x) = \frac{-1}{(1+x)^2}$; $y_2(0) = \frac{-1}{(1+0)^2} = \boxed{-1}$

$y_3(x) = \frac{+2}{(1+x)^3}$; $y_3(0) = \frac{+2}{(1+0)^3} = \boxed{2}$

$y_4(x) = \frac{-6}{(1+x)^4}$; $y_4(0) = \frac{-6}{(1+0)^3} = \boxed{-6}$

Sub in eq^n = 1

$$y(x) = x\cdot 1 + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6)$$

$$y(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

③ Using Maclaurin's series P.T $\sqrt{1+\sin 2x} = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$

Maclaurin's series is given by

$$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \frac{x^4}{4!}y_4(0)$$

$$y(x) = \sqrt{1+\sin 2x}$$
$$= \sqrt{\cos^2 x + \sin^2 x + 2\cos x \sin x}$$
$$= \sqrt{(\cos x + \sin x)^2}$$

$y(x) = \cos x + \sin x \quad ; \quad y(0) = \cos(0) + \sin(0) = 1 + 0 = 1$

$y_1(x) = -\sin x + \cos x \quad ; \quad y_1(0) = -\sin(0) + \cos(0) = 0 + 1 = 1$

$y_2(x) = -\cos x - \sin x \quad ; \quad y_2(0) = -\cos(0) - \sin(0) = -1 - 0 = -1$

$y_3(x) = \sin x - \cos x \quad ; \quad y_3(0) = \sin(0) - \cos(0) = 0 - 1 = -1$

$y_4(x) = \cos x + \sin x \quad ; \quad y_4(0) = \cos(0) + \sin(0) = 1 + 0 = 1.$

Sub in eq^n ①

$$y(x) = 1 + x \cdot 1 + \frac{x^2}{2}(-1) + \frac{x^3}{6}(-1) + \frac{x^4}{4!}(1) \ldots$$

$$y(x) = 1 + x - \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24}$$

④ Expand $\log(\sec x)$ by Maclaurin's series upto the term containing x^6.

Maclaurin's series is given by

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots \rightarrow ①$$

Let $y(x) = \log(\sec x) \quad ; \quad y(0) = \log(\sec 0) = \log 1 = \boxed{0}$

$y_1(x) = \frac{1}{\cancel{\sec x}} \cancel{\sec x} \tan x \quad ; \quad y_1(0) = \tan(0) = \boxed{0}$

$y_2(x) = \sec^2 x \quad ; \quad y_2(0) = \sec^2(0) = 1^2 = \boxed{1}$

$= 1 + \tan^2 x$

$= 1 + y_1^2$

$y_3 = 0 + 2 y_1 y_2 \quad ; \quad y_3(0) = 2 y_1(0) y_2(0)$

$= 2 \times 0 \times 1$

$= \boxed{0}$

$y_4 = 2[y_1 y_3 + y_2 y_2] \quad ; \quad y_4(0) = 2[0 + 1^2] = \boxed{2}$

$= 2[y_1 y_3 + y_2^2]$

$y_5 = 2[y_1 y_4 + y_3 y_2 + 2 y_2 y_3] \quad ; \quad y_5(0) = 2[0 + 0] = 0$

$= 2[y_1 y_4 + 3 y_2 y_3]$

$y_6 = 2[y_1 y_5 + y_4 y_2 + 3 y_2 y_4 + y_3 y_3] \quad ; \quad y_6(0) = 2[0 + (2\times1) + 3(2\times1) + 0]$

$y_6 = 2[2 + 6]$

$y_6 = \boxed{16}$

Substitute in eqⁿ ①

$$y(x) = \frac{x^2}{2!}\times 1 + \frac{x^4}{4!}\times 2 + \frac{x^6}{6!}\times 16$$

$$= \frac{x^2}{2} + \frac{x^4\times 2}{24} + \frac{x^6}{720}\times 16$$

$$= \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45}$$

⑤ Expand $\log(1+\cos x)$ by Maclaurin's series upto the term containing x^4

Maclaurin's series is given by

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots \rightarrow ①$$

Let $y(x) = \log(1+\cos x)$; $y(0) = \log(1+\cos 0) = \boxed{\log 2}$

$y_1(x) = \frac{1}{1+\cos x} \times -\sin x$; $y_1(0) = \frac{-\sin(0)}{1+\cos(0)} = \boxed{0}$

i.e $y_1(x) = \frac{-\cancel{2}\sin x/2 \cdot \cancel{\cos x/2}}{\cancel{2}\cos^{\cancel{2}} x/2}$

$y_1(x) = -\tan x/2$

$y_2(x) = -\sec^2 x/2 \cdot 1/2$; $y_2(0) = -1/2 \sec^2(0) = -1/2\times 1 = \boxed{-1/2}$

i.e. $y_2(x) = -1/2(1+\tan^2 x/2)$

$= -1/2(1+y_1^2)$

$2y_2 = -1 - y_1^2$

$2y_3 = 0 - 2y_1y_2$; $2y_3(0) = -2[0\times -1/2] = 0$

$\cancel{2}y_3 = -\cancel{2}y_1y_2$

$y_3 = -y_1y_2$.

$y_4 = -y_1y_3 + y_2y_2$; $y_4(0) = -[0+(-1/2)^2] = -1/4$.

$y_4 = -y_1y_3$

Sub in eqⁿ ①

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$$y(x) = \log 2 + \frac{x^2}{2!}\left(-\frac{1}{2}\right) + \frac{x^4}{4!}\left(\frac{-1}{4}\right)$$

$$= \log 2 - \frac{x^2}{4} - \frac{x^4}{96}$$

⑥ Expand $e^{\sin x}$ by Maclaurin's series expansion upto the term containing x^4.

Maclaurin's series is given by

$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \dots \rightarrow ①$$

Let $y(x) = e^{\sin x}$; $y(0) = e^{\sin(0)} = e^0 = \boxed{1}$

$y_1(x) = e^{\sin x} \cdot \cos x$; $y_1(0) = e^{\sin(0)} \cdot \cos(0) = e^0 \cdot 1 = \boxed{1}$

i.e $y_1 = y \cos x$

$y_2(x) = -y \sin x + \cos x \, y_1$; $y_2(0) = -1 \times 0 + 1 \times 1 = \boxed{1}$

$y_3(x) = -[y \cos x + \sin x \, y_1] + [\cos x \, y_2 - y_1 \sin x]$

i.e $= -y \cos x - \sin x \, y_1 + \cos x \, y_2 - y_1 \sin x$

$y_3 = -y \cos x - 2 y_1 \sin x + y_2 \cos x$; $y_3(0) = (-1 \times 1) - (2 \times 1 \times 0) + (1 \times 1)$

$y_3 = -1 - 0 + 1 = \boxed{0}$

$y_4 = -[-y \sin x + \cos x \cdot y_1] - 2[y_1 \cos x + \sin x \cdot y_2] + [-y_2 \sin x + \cos x \cdot y_3]$

$y_4(0) = -[(0+1)] - 2(1+0) + [0+0]$

$= -1 - 2$

$y_4 = \boxed{-3}$

Sub in ①

$$y(x) = 1 + x(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-3)$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8}$$

⑦ $\log(1+e^x)$

$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) \dots$

$y = \log(1+e^x)$; $y(0) = \log(1+1) = \log 2$

$y_1 = \frac{e^x}{1+e^x}$; $y_1(0) = \frac{1}{1+1} = \frac{1}{2}$

$(1+e^x)y_1 = e^x$

$(1+e^x)y_2 + e^x y_1 = e^x$; $y_2(0) = y_2(1+1) + y_1(1) = e^x$

$2y_2 = 1 - \frac{1}{2}$ $\quad \therefore y_2(0) = \frac{1}{4}$

$(1+e^x)y_2 + e^x y_2 + e^x y_2 + e^x y_1 = e^x$

$(1+e^x)y_3 + 2e^x y_2 + e^x y_1 = e^x$

$y_3(0) = (1+1)y_3(0) + 2e^0\left(\frac{1}{4}\right) + 1(1/2) = 1$

$2y_3(0) + \frac{1}{2} + \frac{1}{2} = 1$

$y_3(0) = 0$

$(1+e^x)y_4 + e^x y_3 + 2[e^x y_3 + e^x y_2] + [e^x y_2 + e^x y_1] = e^x$

$(1+e^x)y_4 + 3e^x y_3 + 3e^x y_2 + e^x y_1 = e^x$

$y_4(0) = \frac{1}{8}$

Sub in ①

$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} \dots \frac{x^4}{142}$

⑧ $\log(1+\sin x)$ upto x^4

$y(x) = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0)$

$y(x) = \log(1+\sin x)$; $y(0) = \log(1+0) = \log 1 = 0$

$y_1(x) = \frac{1}{1+\sin x}(\cos x)$; $y_1(0) = \frac{1}{1+0} = 1$

$y_2(x) = \frac{(1+\sin x)(-\sin x) - \cos x\,(0+\cos x)}{(1+\sin x)^2}$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-(\sin x + \sin^2 x + \cos^2 x)}{(1+\sin x)^2}$$

$$= \frac{-(\sin x + 1)}{(1+\sin x)^2} \quad ; \; y_2 = \frac{-1}{1+\sin 0} = \frac{-1}{1}$$

$$y_2(0) = -1$$

$$y_3 = \frac{(1+\sin x)0 + (-1)(0+\cos x)}{(1+\sin x)^2} \quad ; \; y_3(0) = \frac{\pm 1}{1+0} = 1$$

diff

$$y_4 = \frac{(1+\sin x)^2(-\sin x) - \cos x \; 2(1+\sin x)\cos x}{(1+\sin x)^4}$$

$$= \frac{1+\sin x\left[(1+\sin x)(-\sin x) - 2\cos^2 x(1)\right]}{(1+\sin x)^2}$$

$$= \frac{-\sin x - 2\cos^2 x}{1+\sin x}$$

$$= \frac{-\sin(0) - 2\cos^2(0)}{1+\sin 0}$$

$$y_4(0) = \frac{0-2(1)}{1+0} = \frac{-2}{1}$$

$$y_4(0) = -2$$

Sub in ①

$$y(x) = \log 1 + x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12}$$

⑨ $\sqrt{1+\cos 2x} = \sqrt{2}$

$$y(x) = \sqrt{1+\cos 2x} = \sqrt{1+2\cos^2 x - 1} = \sqrt{2\cos^2 x} = \sqrt{2}\cos x$$

$$y(0) = \sqrt{2}\cos x = \sqrt{2}$$

$$y_1(x) = \sqrt{2}(-\sin x) = -\sqrt{2}\sin x$$

$$y_1(0) = -\sqrt{2}\sin(0) = 0$$

$y_2(x) = -\sqrt{2}\cos x = -\sqrt{2}\cos x$
$y_2(0) = -\sqrt{2}\cos(0) = -\sqrt{2}$

$y_3(x) = \frac{d}{dx}(\sqrt{2}\cos x) = \sqrt{2}\sin x$

$y_3(0) = \sqrt{2}\sin(0) = 0$

$y_4(x) = \frac{d}{dx}(\sqrt{2}\sin x) = \sqrt{2}\cos x$

$y_4(0) = \sqrt{2}\cos(0) = \sqrt{2}$
$y_4(0) = \sqrt{2}$

Sub in ①
$y(x) = \sqrt{2} + 0 + \frac{x^2}{2!}(-\sqrt{2}) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(\sqrt{2}) + \ldots$

$y(x) = \sqrt{2} - \frac{x^2}{2}(-\sqrt{2}) + \frac{x^4}{24}(\sqrt{2})$

$y(x) = \sqrt{2}\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots\right]$

⑩ $e^x \cos x$ upto x^4

$y(x) = y(0) + xy_1 + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) \ldots$

$y(x) = e^x \cos x$; $y(0) = e^0 \cos(0) = 1$
$y_1(x) = e^x(-\sin x) + \cos x(e^x)$; $y_1(0) = e(-e^0 \sin 0) + \cos 0(e^0)$
$= -e^x \sin x + e^x \cos x$ $y_1(0) = 1$

$y_2(x) = -[e^x \cos x + \sin x e^x] + e^x(-\sin x) + \cos e^x$
$= -e^x \cos x - \sin x e^x - e^x \sin x + e^x \cos x$
$= -2\sin x(e^x)$
$y_2(0) = -2\sin(0)(1)$
$y_2(0) = 0$

$y_3(x) = -2[\sin x\, e^x + e^x \cos x]$
$y_3(0) = -2[0+1]$
$y_3(0) = -2$

$y_4(x) = -2[(\sin e^x + e^x \cos x) + (e^x \cos x + (-\sin x\, e^x))]$
$= -2[e^x \sin x + e^x \cos x + e^x \cos x - e^x \sin x]$
$y_4(0) = -2[0+1+1-0]$
$y_4(0) = -4$

Sub in ①
$$y(x) = 1 + x + \frac{x^2}{2}(0) + \frac{x^3}{6}(-2) + \frac{x^4}{24}(-4)$$

$$y(x) = 1 + x - \frac{x^3}{3} - \frac{x^4}{6}$$

(ii) $\tan x$ upto x^5
$$y(x) = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \cdots$$

$y(x) = \tan x \quad ; y(0) = 0$
$y_1(x) = \sec^2 x \quad ; y_1(0) = 1$
$= 1 + \tan^2 x$
$y_1 = 1 + y^2$
$y_2(x) = 2y \cdot y_1 \quad ; y_2(0) = 0$
$y_3(x) = [2yy_2 + 2y_1^2] \quad ; y_3(0) = 2[0+1] = 2$
$y_4(x) = 4y_1y_3 + 4y_2^2 + 2yy_4 + 2y_3y_2 + 4y_2y_3$
$y_4(0) = 4(1)(2) + 4(0)^2 + 2(1)(0) + 2(2)(0) + 4(0)(2)$
$= 8 + 0 + 0 + 0 + 8$
$y_4(0) = 16$
$$y(x) = x + \frac{2x^3}{6} + \frac{16x^5}{120}$$

$$y(x) = x + \frac{x^3}{3} + \frac{2x^5}{15}$$

Indeterminate Forms : L' Hospital rule

If $f(x)$ and $g(x)$ are two functions such that

(i) $\lim_{x \to a} f(x) = 0$ & $\lim_{x \to a} g(x) = 0$

i.e $f(a) = 0$ & $g(a) = 0$

(ii) $f'(x)$ & $g'(x)$ exists with $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Evaluation of limits of the form $1^\infty, 0^0, \infty^0, 0^\infty$ etc.

Working rule.

Step ①: Let $k = \lim_{x \to a} \{f(x)\}^{g(x)}$

②: Take log on b.s so that $\log_e k = \lim_{x \to a} \log \{f(x)\}^{g(x)}$

$\Rightarrow \log_e k = \lim_{x \to a} g(x) \cdot \log \{f(x)\}$

③: Reduce the RHS of above expression to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form to apply LHR.

④ If the limit evaluated in the RHS is l (say) then we have $\log_e k = l$ (say)

$\Rightarrow \boxed{k = e^l}$ is the required limit

Standard limits

① $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

② $\lim_{x \to 0} \frac{x}{\sin x} = 1.$

③ $\lim_{x \to 0} \frac{\tan x}{x} = 1$

④ $\lim_{x \to 0} \frac{x}{\tan x} = 1.$

Note : ① $e^0 = 1$

② $e^{\infty} = \infty$, $\log \infty = \infty$

③ $e^{-\infty} = 0$, $\log 0 = -\infty$

④ $1/0 = \infty$

Problems

Evaluate the following Indeterminate forms:-

① $\lim_{x \to 0} \left(\frac{1}{x}\right)^{2\sin x}$

Let $k = \lim_{x \to 0} \left(\frac{1}{x}\right)^{2\sin x}$ $\quad (\infty^0)$

$\log_e k = \lim_{x \to 0} \log_e \left(\frac{1}{x}\right)^{2\sin x}$

$= \lim_{x \to 0} 2\sin x \cdot \log\left(\frac{1}{x}\right)$

$= \lim_{x \to 0} \frac{2\log\left(\frac{1}{x}\right)}{1/\sin x}$

$\log k = \lim_{x \to 0} \frac{2(\log 1 - \log x)}{\operatorname{cosec} x}$ $\quad \left(\frac{\infty}{\infty}\right)$

Apply LHR

$= \lim_{x \to 0} \frac{2\left\{0 - \frac{1}{x}\right\}}{-\operatorname{cosec} x \cdot \cot x}$

$= \lim_{x \to 0} -\frac{2}{x} \times \sin x \times \tan x$ $\quad \left(\frac{0}{0}\right)$

$= \left(\lim_{x \to 0} \frac{\sin x}{x}\right) \times \left(\lim_{x \to 0} 2\tan x\right)$

$= 1 \cdot (2 \times 0)$

$\log_e k = 0$

$\boxed{k = e^0 = 1}$

② $\lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

Let $k = \lim_{x \to a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ $\quad (1^{\infty})$

$$\log k = \lim_{x \to a} \log\left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

$$= \lim_{x \to a} \tan\left(\frac{\pi x}{2a}\right) \cdot \log\left(2 - \frac{x}{a}\right)$$

$$= \lim_{x \to a} \frac{\log\left(2 - \frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left(\frac{0}{0}\right)$$

Apply LHR.

$$\log_e k = \lim_{x \to a} \frac{\frac{1}{\left(2 - \frac{x}{a}\right)} \times \left(0 - \frac{1}{a}\right)}{-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right) \times \frac{\pi}{2a} \cdot 1}$$

$$= \lim_{x \to a} \frac{-1}{a} \cdot \frac{1}{\left(2 - \frac{x}{a}\right)} \times \frac{2a}{\pi} \times -\sin^2\left(\frac{\pi x}{2a}\right)$$

$$= \lim_{x \to a} \frac{2}{\pi} \frac{\sin^2\left(\frac{\pi x}{2a}\right)}{\left(2 - \frac{x}{a}\right)}$$

$$= \frac{2}{\pi} \cdot \frac{\sin^2\left(\frac{\pi}{2}\right)}{\left(2 - \frac{a}{a}\right)}$$

$$= \frac{2}{\pi} \cdot \frac{1^2}{(2-1)}$$

$$\log_e k = \frac{2}{\pi}$$

$$\boxed{k = e^{2/\pi}}$$

** ③ $\lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$

Let $k = \lim_{x \to 0} \left(\frac{a^x + b^x}{2}\right)^{1/x}$ $\qquad (1^\infty)$

$$\log_e k = \lim_{x \to 0} \log_e \left(\frac{a^x + b^x}{2}\right)^{1/x}$$

$$= \lim_{x \to 0} \frac{1}{x} \cdot \log_e \left(\frac{a^x + b^x}{2}\right)$$

$$= \lim_{x \to 0} \frac{\log\left(\frac{a^x + b^x}{2}\right)}{x} \qquad \left(\frac{0}{0}\right)$$

Apply LHR.

$$\log_e k = \lim_{x\to 0} \frac{\left(\frac{1}{\frac{a^x+b^x}{2}}\right)\{a^x \cdot \log a + b^x \cdot \log b\} \cdot \frac{1}{2}}{1}$$

$$= \frac{1}{(a^0+b^0)}\{a^0 \cdot \log a + b^0 \log b\}$$

$$= \frac{1}{1+1}\{\log a + \log b\}$$

$$= \frac{1}{2}\log(ab)$$

$$= \log(ab)^{1/2}$$

$$\log_e k = \log_e \sqrt{ab}$$

$$\boxed{k = \sqrt{ab}}$$

(4) $\lim_{x\to 0}\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{1/x}$

$$\text{Let } k = \lim_{x\to 0}\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{1/x} \quad (1^\infty)$$

$$\log_e k = \lim_{x\to 0} \log_e\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{1/x}$$

$$= \lim_{x\to 0}\frac{1}{x}\cdot \log_e\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{1/x}$$

$$= \lim_{x\to 0}\frac{\log\left(\frac{a^x+b^x+c^x+d^x}{4}\right)}{x} \quad \left(\frac{0}{0}\right)$$

Apply LHR.

$$\log_e k = \lim_{x\to 0}\frac{\left(\frac{1}{\frac{a^x+b^x+c^x+d^x}{4}}\right)\{a^x \cdot \log a + b^x \cdot \log b + c^x \log c + d^x \log d\} \cdot \frac{1}{4}}{1}$$

$$= \frac{1}{(a^0+b^0+c^0+d^0)}\{a^0 \log a + b^0 \log b + c^0 \log c + d^0 \log d\}$$

$$= \frac{1}{1+1+1+1}\{\log a + \log b + \log c + \log d\}$$

$$= \frac{1}{4}\log(abcd)$$

$= \log (abcd)^{1/4}$

$\log_e k = \log_e (abcd)^{1/4}$

$\boxed{k = (abcd)^{1/4}}$

(5) $\lim_{x\to 0} (\cos x)^{1/x^2}$

let $k = \lim_{x\to 0} (\cos x)^{1/x^2}$ (1^∞)

$\log_e k = \lim_{x\to 0} \log (\cos x)^{1/x^2}$

$\log_e k = \lim_{x\to 0} \frac{1}{x^2} \frac{\log_e(\cos x)}{x^2}$ $\left(\frac{0}{0}\right)$

Apply LHR.

$\log_e k = \lim_{x\to 0} \frac{\frac{1}{\cos x} \times -\sin x}{2x}$

$= \lim_{x\to 0} \frac{-\tan x}{2x}$ $\left(\frac{0}{0}\right)$

$= \frac{-1}{2}\left(\lim_{x\to 0} \frac{\tan x}{x}\right)$

$= \frac{-1}{2} \times 1.$

$\log_e k = -\frac{1}{2}$

$k = e^{-1/2} = 1/e^{1/2}$

$\boxed{k = \frac{1}{\sqrt{e}}}$

** (6) $\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x}$

let $k = \left[\lim_{x\to 0} \left(\frac{\tan x}{x}\right)\right]^{1/x}$ (1^∞)

$\log_e k = \lim_{x\to 0} \log \left(\frac{\tan x}{x}\right)^{1/x}$

* Note : $\lim_{x\to a} (A \pm B) \neq \lim_{x\to a} A \pm \lim_{x\to a} B$

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$$= \lim_{x\to 0} \frac{1}{x} \cdot \log\left(\frac{\tan x}{x}\right)$$

$$= \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x} \qquad \left(\frac{0}{0}\right)$$

Apply LHR.

$$\log_e k = \lim_{x\to 0} \frac{\frac{1}{\frac{\tan x}{\cancel{x}}}\left\{\frac{x \cdot \sec^2 x - \tan x \cdot 1}{x^2}\right\}}{1}$$

$$= \lim_{x\to 0} \left\{\frac{x \cdot \sec^2 x - \tan x}{x^2}\right\} \qquad \left(\frac{0}{0}\right) \quad \left[\because \lim_{x\to 0} \frac{x}{\tan x} = 1\right]$$

Apply LHR.

$$\log_e k = \lim_{x\to 0} \left\{\frac{x \cdot 2\sec^2 x \cdot \tan x + \cancel{\sec^2 x \cdot 1} - \cancel{\sec^2 x}}{2x}\right\}$$

$$= \lim_{x\to 0} \left\{\frac{\cancel{2x} \sec^2 x \cdot \tan x}{\cancel{2x}}\right\}$$

$$= \sec^2(0) \cdot \tan(0)$$

$$\log_e k = 0.$$

$$\boxed{k = e^0 = 1}$$

(7) $\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{1/x^2}$

$$\text{Let } k = \left[\lim_{x\to 0} \frac{\sin x}{x}\right]^{1/x^2} \qquad (1^\infty)$$

$$\log_e k = \lim_{x\to 0} \log\left(\frac{\sin x}{x}\right)^{1/x^2}$$

$$= \lim_{x\to 0} \frac{1}{x^2} \cdot \log\left(\frac{\sin x}{x}\right)$$

$$= \lim_{x\to 0} \frac{\log\left(\frac{\sin x}{x}\right)}{x^2} \qquad \left(\frac{0}{0}\right)$$

Apply LHR.

$$\log_e k = \lim_{x\to 0} \frac{\frac{1}{\frac{\sin x}{x}}\left\{\frac{x \cdot \cos x - \sin x \cdot 1}{x^2}\right\}}{2x}$$

$$= \lim_{x\to 0} \left\{ \frac{x\cos x - \sin x}{2x^3} \right\} \quad \frac{0}{0} \quad \left[\because \lim_{x\to 0} \frac{x}{\sin x} = 1 \right]$$

Apply LHR.

$$\log_e k = \lim_{x\to 0} \left\{ \frac{\cancel{-x.\sin x} + \cancel{\cos x.1} - \cancel{\cos x}}{6x^2} \right\}$$

$$= \lim_{x\to 0} \left\{ \frac{-\cancel{x}\sin x}{6x^{\cancel{2}}} \right\}$$

$$= \frac{-1}{6} \lim_{x\to 0} \frac{\sin x}{x}$$

$$= \frac{-1}{6} \times 1.$$

$$\log_e k = \frac{-1}{6}$$

$$\boxed{k = e^{-1/6}}$$

⑧ $\lim_{x\to 0} (a^x + x)^{1/x}$

Let $K = \lim_{x\to 0} (a^x + x)^{1/x}$ $\quad (1^\infty)$

Take log on b.s

$$\log_e K = \lim_{x\to 0} \log (a^x + x)^{1/x}$$

$$\log_e K = \lim_{x\to 0} \frac{1}{x} \log (a^x + x)$$

$$\log_e K = \lim_{x\to 0} \frac{\log (a^x + x)}{x} \quad \left(\frac{0}{0}\right)$$

Applying LHR

$$\log_e k = \lim_{x\to 0} \frac{\frac{1}{(a^x + x)} \cdot [a^x . \log a + 1)}{1}$$

$$= \lim_{x\to 0} \frac{(a^x . \log a + 1)}{(a^x + x)}$$

$$\log_e k = \frac{a^0 . \log a + 1}{a^0 + 0}$$

$$\log_e k = \frac{\log a + 1}{1.}$$

$\log_e k = \log a + 1$

$k = a + e \Rightarrow k = e^{\log a + 1}$

$\Rightarrow e^{\log a} \cdot e^{1} \Rightarrow \boxed{k = ae}$

(9) $\lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$

Let $k = \lim_{x \to 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$ $= \left(\frac{3}{3}\right)^{1/0} = 1^{\infty}$

Take log on b.s

$\log_e k = \lim_{x \to 0} \log\left(\frac{a^x + b^x + c^x}{3}\right)^{1/x}$

$= \lim_{x \to 0} \frac{1}{x} \log\left(\frac{a^x + b^x + c^x}{3}\right)$

$= \lim_{x \to 0} \frac{\log\left(\frac{a^x + b^x + c^x}{3}\right)}{x}$ $\left(\frac{0}{0}\right)$

Apply LHR.

$\log_e k = \lim_{x \to 0} \frac{\frac{1}{\frac{a^x + b^x + c^x}{3}} \cdot \frac{(a^x \cdot \log a + b^x \log b + c^x \cdot \log c)}{3}}{1}$

$= \lim_{x \to 0} \frac{a^x \cdot \log a + b^x \log b + c^x \log c}{a^x + b^x + c^x}$

$= \frac{a^0 \log a + b^0 \log b + c^0 \log c}{a^0 + b^0 + c^0}$

$= \frac{\log a + \log b + \log c}{3}$

$= \frac{1}{3} \log(abc)$

$k = (abc)^{1/3}$

(10) $\lim_{x \to 0} (\cos x)^{\cot^2 x}$

Let $k = \lim_{x \to 0} (\cos x)^{\cot^2 x}$ (1^{∞})

Take log on b.s

$\log_e k = \lim_{x \to 0} \log(\cos x)^{\cot^2 x}$

$$= \lim_{x\to 0} \cot^2 x \log(\cos x)$$

$$= \lim_{x\to 0} \frac{\log \cos x}{\tan^2 x} \quad \left(\frac{0}{0}\right)$$

Apply LHR.

$$\log_e k = \lim_{x\to 0} \frac{\log(\cos x)}{\cot^2 x}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2\cot x \cdot -\operatorname{cosec} x \cot x \cdot 2\tan x \sec^2 x}$$

$$= \lim_{x\to 0} \frac{\frac{-\sin x}{\cos x}}{2 \cdot \tan x \cdot \sec^2 x}$$

$$= \lim_{x\to 0} \frac{-\cancel{\tan x}}{2\cancel{\tan x} \cdot \sec^2 x}$$

$$= \lim_{x\to 0} -\frac{1}{2}(\cos^2 x)$$

$$= -\frac{1}{2}(1)^2$$

$$\log_e k = -1/2$$

$$k = e^{-1/2}$$

$$k = \frac{1}{e^{1/2}}$$

$$k = \frac{1}{\sqrt{e}}$$

(11) $\lim_{x\to \frac{\pi}{2}} (\tan x)^{\cos x}$

Let $k = \lim_{x\to \frac{\pi}{2}} (\tan x)^{\cos x} \quad = (\infty^0)$

$$\log_e k = \lim_{x\to \frac{\pi}{2}} \cos x \cdot \log(\tan x)$$

$$\log_e k = \lim_{x\to \pi/2} \frac{\log(\tan x)}{1/\cos x}$$

$$= \lim_{x\to \pi/2} \frac{\log(\tan x)}{\sec x} \quad \left(\frac{\infty}{\infty}\right)$$

Apply LHR

$$\log_e k = \lim_{x\to \pi/2} \frac{\frac{1}{\tan x}(\sec^2 x)}{\sec x \cdot \tan x}$$

$$= \lim_{x\to\pi/2} \frac{\frac{\sec^2 x}{\tan x}}{\cancel{\sec x}\cdot \tan x}$$

$$= \lim_{x\to\pi/2} \frac{\sec x}{\tan^2 x}$$

Apply LHR.

$$\log_e k = \lim_{x\to\pi/2} \frac{2\sec x \tan x}{2\tan x \cdot \sec^2 x}$$

$$= \lim_{x\to\pi/2} \frac{1}{2} \cdot \cos x$$

$k = e^0$

$k = 1.$

(12) $\lim_{x\to\pi/2} (\sin x)^{\tan x}$

Let $k = (\sin x)^{\tan x}$

$\lim_{x\to\pi/2} = \log(\sin x) \tan x$

$\lim_{x\to\pi/2} = \tan x \log(\sin x)$

$$= \frac{1/\sin x}{1/\tan x}$$

$$= \frac{1/\sin x}{\cot x}$$

$$= \frac{\frac{1}{\sin x}(\cos x)}{-\operatorname{cosec} x}$$

$$\lim_{x\to\pi/2} = \frac{\frac{\cos x}{\sin x}}{\frac{1}{-\sin^2 x}}$$

$\lim_{x\to\pi/2} = \cos x - \sin x$

$= -\sin x \cdot \cos x$

$= -\sin \pi/2 \times \cos \pi/2$

$= -1 \times 0$

$\log_e k = 0$

$k = e^0 = \boxed{k = 1}$

→ Partial differentiation

Defn: The process of finding Partial derivatives is called Partial differentiation.
If 'u' is a function of two independent variables (x, y) i.e $(u = f(x,y))$ then the derivative of u w.r.t x when x varies and y remains constant is called Partial derivative of u w.r.t x and is denoted by $\frac{\partial u}{\partial x}$ or u_x

Similarly, to obtain $\frac{\partial u}{\partial y}$ or u_y, y varies and x remains constant.

Ex: i) $\frac{\partial}{\partial x}(2x^2+y) = 2\frac{\partial}{\partial x}(x^2) + y\frac{\partial}{\partial x}(1)$

$= 2 \cdot 2x + y \times 0$

$= 4x + 0$

$= 4x.$

ii) $\frac{\partial}{\partial y}(2x^2+y) = 0 + 1 = 1$

iii) $\frac{\partial}{\partial x}(x^3y^2) = y^2 \cdot \frac{\partial}{\partial x}(x^3)$

$= y^2 \cdot 3x^2$

$= 3x^2y^2.$

iv) $\frac{\partial}{\partial y}(x^3y^2) = x^3 \cdot 2y$

Note: ① First order Partial derivatives of 'u' w.r.t x, y are $\frac{\partial u}{\partial x}$ (or) u_x and $\frac{\partial u}{\partial y}$ (or) u_y.

② Second order Partial derivatives of 'u' w.r.t x, y are

i) $\frac{\partial^2 u}{\partial x^2}$ (or) u_{xx}

ii) $\frac{\partial^2 u}{\partial x \partial y}$ (or) u_{xy}

iii) $\frac{\partial^2 y}{\partial y \partial x}$ (or) u_{yx}

iv) $\frac{\partial^2 x}{\partial y^2}$ (or) u_{yy}

③ $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

④ If u is a function of z & z is a function of x, y then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}$ & $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y}$

Problems

① If $u = x^2 y + y^2 z + z^2 x$, then show that $u_x + u_y + u_z = (x+y+z)^2$.

$u = x^2 y + y^2 z + z^2 x$

$\frac{\partial u}{\partial x} = y \cdot 2x + 0 + z^2 \cdot 1$

i.e $\boxed{u_x = 2xy + z^2} \rightarrow$ ①

$\frac{\partial u}{\partial y} = x^2 \cdot 1 + z \cdot 2y + 0$

i.e $\boxed{u_y = x^2 + 2yz} \rightarrow$ ②

$\frac{\partial u}{\partial z} = 0 + y^2 \cdot 1 + x \cdot 2z$

i.e $\boxed{u_z = y^2 + 2xz} \rightarrow$ ③

Add ①, ② and ③

$$u_x + u_y + u_z = 2xy + z^2 + x^2 + 2yz + y^2 + 2xz$$
$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$
$$= (x+y+z)^2$$

*③ If $z = e^{ax+by} f(ax-by)$ Show that $b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz$

$$z = e^{ax+by} f(ax-by)$$

$$\frac{\partial z}{\partial x} = e^{ax+by} \cdot f'(ax-by) \cdot \frac{\partial}{\partial x}(ax-by) + f(ax-by)\, e^{ax+by} \cdot \frac{\partial}{\partial x}(ax+by)$$

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by)\, a + f(ax-by) e^{ax+by} \cdot a$$

$$\frac{\partial z}{\partial x} = a e^{ax+by} f'(ax-by) + az$$

Multiply on b.s by b.

$$\boxed{b\frac{\partial z}{\partial x} = abe^{ax+by} f'(ax-by) + abz} \rightarrow ①$$

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by)\, e^{ax+by} \cdot b$$

$$\frac{\partial z}{\partial y} = -be^{ax+by} f'(ax-by) + bz.$$

Multiply on b.s by a.

$$\boxed{a\frac{\partial z}{\partial y} = -abe^{ax+by} f'(ax-by) + abz} \rightarrow ②$$

adding ① & ②

$$b\frac{\partial z}{\partial x} + a\frac{\partial z}{\partial y} = 2abz.$$

→ Jacobians.

① If u, v are functions of 2 independent variables x, y then.

$$J\left(\frac{u,v}{x,y}\right) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

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(3) If u, v, w are functions of 3 independent variables x, y, z then.

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Problems:

(1) If $u = e^x \cos y$; $v = e^x \sin y$, find $\frac{\partial(u,v)}{\partial(x,y)}$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \rightarrow (1)$$

Consider $u = e^x \cos y$

$\frac{\partial u}{\partial x} = e^x \cos y$; $\frac{\partial u}{\partial y} = -e^x \sin y$

Consider $v = e^x \sin y$

$\frac{\partial v}{\partial x} = e^x \sin y$; $\frac{\partial v}{\partial y} = e^x . \cos y$

$(e^x)^2 = e^{2x}$

Sub in (1)

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x . \cos y \end{vmatrix}$$

$$= (e^x \cos y)(e^x \cos y) - (-e^x \sin y)(e^x \sin y)$$

$$= e^{2x} . \cos^2 y + e^{2x} . \sin^2 y$$

$$= e^{2x} (\cos^2 y + \sin^2 y)$$

$$= \underline{\underline{e^{2x}}}$$

(2) Find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ where $u = x^2 + y^2 + z^2$, $v = xy + yz + zx$, $w = x + y + z$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \rightarrow (1)$$

Consider $u = x^2 + y^2 + z^2$

$$\frac{\partial u}{\partial x} = 2x + 0 + 0 = 2x.$$

$$\frac{\partial u}{\partial y} = 0 + 2y + 0 = 2y$$

$$\frac{\partial u}{\partial z} = 0 + 2z + 0 = 2z.$$

Consider $v = xy + yz + zx$

$$\frac{\partial v}{\partial x} = y.1 + z.1 = y + z$$

$$\frac{\partial v}{\partial y} = x.1 + z.1 + 0 = x + z$$

$$\frac{\partial z}{\partial y} = 0 + y.1 + x.1 = y + x$$

Consider $w = x + y + z$

$$\frac{\partial w}{\partial x} = x = 1.$$

$$\frac{\partial w}{\partial y} = y = 1$$

$$\frac{\partial w}{\partial z} = z = 1.$$

Sub in ①

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \overset{+}{2x} & \overset{-}{2y} & \overset{+}{2z} \\ y+z & x+z & y+x \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2x\{(x+z) - (y+x)\} - 2y\{(y+z) - (y+x)\} + 2z\{(y+z) - (x+z)\}$$

$$= 2x\{z-y\} - 2y\{z-x\} + 2z\{y-x\}$$

$$= \cancel{2xz} - \cancel{2xy} - \cancel{2yz} + \cancel{2yx} + \cancel{2zy} - \cancel{2xz}.$$

$$= 0$$

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③ If $u = x + 3y^2 - z^3$, $v = 4x^2yz$, $w = 2z^2 - xy$, Evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $[1,-1,0]$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \rightarrow ①$$

Consider $u = x + 3y^2 - z^3$

$\frac{\partial u}{\partial x} = 1$; $\frac{\partial u}{\partial y} = 6y$; $\frac{\partial u}{\partial z} = -3z^2$

Consider $v = 4x^2yz$.

$\frac{\partial v}{\partial x} = 4yz \cdot 2x$; $\frac{\partial v}{\partial y} = 4x^2z \cdot 1$; $\frac{\partial v}{\partial z} = 4x^2y \cdot 1$.

$= 8xyz$; $= 4x^2z$; $= 4x^2y$

Consider $w = 2z^2 - xy$.

$\frac{\partial w}{\partial x} = -y \cdot 1$; $\frac{\partial w}{\partial y} = 0 - x \cdot 1$; $\frac{\partial w}{\partial z} = 4z - 0$

$= -y$; $= x$; $= 4z$

Sub in ①

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \overset{+}{1} & \overset{-}{6y} & \overset{+}{-3z^2} \\ 8xyz & 4x^2z & 4x^2y \\ -y & x & 4z \end{vmatrix}$$

$$\text{at } [1,-1,0] = \begin{vmatrix} \overset{+}{1} & \overset{-}{-6} & \overset{+}{0} \\ 0 & 0 & -4 \\ 1 & 1 & 0 \end{vmatrix}$$

$= 1(0-4) - (-6)(0+4) + 0(0+4)$

$= -4 + 24$

$= 20$.

④ If $u = \frac{xy}{z}$, $v = \frac{yz}{x}$, $w = \frac{zx}{y}$ then S.T $J = \left(\frac{u,v,w}{x,y,z}\right) = 4$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \rightarrow ①$$

Consider $u = \frac{xy}{z}$

$\frac{\partial u}{\partial x} = \frac{y}{z} \cdot 1$; $\frac{\partial u}{\partial y} = \frac{x}{z} \cdot 1$; $\frac{\partial u}{\partial z} = xy\left\{\frac{-1}{z^2}\right\}$

$= \frac{y}{z}$ $\quad = \frac{x}{z}$ $\quad = \frac{-xy}{z^2}$

Consider $v = \frac{yz}{x}$

$\frac{\partial v}{\partial x} = yz\left\{\frac{-1}{x^2}\right\}$; $\frac{\partial v}{\partial y} = \frac{z}{x} \cdot 1$; $\frac{\partial v}{\partial z} = \frac{y}{x} \cdot 1$

$= \frac{-yz}{x^2}$ $\quad = \frac{z}{x}$ $\quad = \frac{y}{x}$

Consider $w = \frac{zx}{y}$

$\frac{\partial w}{\partial x} = \frac{z}{y}$; $\frac{\partial w}{\partial y} = zx \cdot \left\{\frac{-1}{y^2}\right\}$; $\frac{\partial w}{\partial z} = \frac{x}{y}$

$= \frac{-zx}{y^2}$

Sub in (1)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} y/z & x/z & -xy/z^2 \\ -yz/x^2 & z/x & y/x \\ z/y & -zx/y^2 & x/y \end{vmatrix}$$

$$= \frac{1}{z^2x^2y^2}\begin{vmatrix} yz & xz & -xy \\ -yz & zx & yx \\ zy & -zx & xy \end{vmatrix}$$

$$= \frac{1}{x^2y^2z^2}\left[yz(x^2yz + x^2yz) - xz(-xy^2z - xy^2z) - xy(\cancel{xyz^2} - \cancel{xyz^2})\right]$$

$$= \frac{1}{x^2y^2z^2}\left[(yz \cdot 2x^2yz) - (xz)(-2xy^2z)\right]$$

$$= \frac{1}{x^2y^2z^2}\left[2x^2y^2z^2 + 2x^2y^2z^2\right]$$

$$= \frac{1}{\cancel{x^2y^2z^2}}\left[4\cancel{x^2y^2z^2}\right]$$

$= 4$

(5) If $x+y+z=u$, $y+z=uv$, $z=uvw$ then evaluate $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \rightarrow (1)$$

Consider, $x+y+z=u$; $y+z=uv$; $z=uvw$

$x+uv=u$ $\qquad y=uv-z$

$\boxed{x=u-uv}$ $\qquad \boxed{y=uv-uvw}$ $\qquad \boxed{z=uvw}$

Consider $x=u-uv$

$\frac{\partial x}{\partial u} = 1-v\cdot 1$; $\frac{\partial x}{\partial v} = 0-u\cdot 1$; $\frac{\partial x}{\partial w} = 0.$

$= 1-v$ $\qquad = -u$

Consider, $y=uv-uvw$.

$\frac{\partial y}{\partial u} = v\cdot 1 - vw\cdot 1$; $\frac{\partial y}{\partial v} = u\cdot 1 - uw\cdot 1$; $\frac{\partial y}{\partial w} = 0-uv\cdot 1$

$= v-vw$ $\qquad = u-uw$ $\qquad = -uv.$

Consider, $z=uvw$.

$\frac{\partial z}{\partial u} = vw$; $\frac{\partial z}{\partial v} = uw$; $\frac{\partial z}{\partial w} = uv.$

Sub in (1).

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \overset{+}{1-v} & \overset{-}{-u} & \overset{+}{0} \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$= (1-v)[(u-uw)(uv)+u^2vw] + u[(v-vw)(uv)+uv^2w] + 0$

$= (1-v)[u^2v - \cancel{u^2vw} + \cancel{u^2vw}] + u[uv^2 - \cancel{uv^2w} + \cancel{uv^2w}]$

$= (1-v)(u^2v) + u^2v^2$

$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$

$= u^2v.$

⑥ If $u = x\cos y\cos z$, $v = x\cos y\sin z$, $w = x\sin y$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = -x^2\cos y$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \rightarrow ①$$

Consider $u = x\cos y\cos z$

$\frac{\partial u}{\partial x} = 1\cos y\cos z$; $\frac{\partial u}{\partial y} = -x\sin y\cos z$; $\frac{\partial u}{\partial z} = x\cos y\sin z$

Consider $v = x\cos y\sin z$

$\frac{\partial v}{\partial x} = \cos y\sin z$; $\frac{\partial v}{\partial y} = -x\sin y\sin z$; $\frac{\partial v}{\partial z} = x\cos y\cos z$

Consider $w = x\sin y$

$\frac{\partial w}{\partial x} = \sin y$; $\frac{\partial w}{\partial y} = x\cos y$; $\frac{\partial w}{\partial z} = 0$

Substitute in eq^n ①

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \overset{+}{\cos y\cos z} & \overset{-}{-x\sin y\cos z} & \overset{+}{x\cos y\sin z} \\ \cos y\sin z & -x\sin y\sin z & x\cos y\cos z \\ \sin y & x\cos y & 0 \end{vmatrix}$$

$= \cos y\cos z\,[0 - x^2\cos^2 y\cos z] + x\sin y\cos z\,[0 - x\sin y\cos y\cos z]$
$\quad - x\cos y\sin z\,[x\cos^2 y\sin z + x\sin^2 y\sin z]$

$= -x^2\cos^3 y\cos^2 z - x^2\sin^2 y\cos y\cos^2 z - x\cos y\sin z(x\sin z)$

$= -x^2\cos^3 y\cos^2 z - x^2\sin^2 y\cos y\cos^2 z - x^2\cos y\sin^2 z$

$= -x^2\cos y\cos^2 z(\cos^2 y + \sin^2 y) - x^2\cos y\sin^2 z$

$= -x^2\cos y\cos^2 z\,(1) - x^2\cos y\sin^2 z$

$= -x^2\cos y(\cos^2 z + \sin^2 z)$

$= -x^2\cos y$

⑦ If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, find $J\left(\frac{x,y,z}{r,\theta,\phi}\right)$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} \rightarrow (1)$$

Consider $x = r\sin\theta\cos\phi$

$\frac{\partial x}{\partial r} = \sin\theta\cos\phi$; $\frac{\partial x}{\partial \theta} = r\cos\phi\cos\theta$; $\frac{\partial x}{\partial \phi} = r\sin\theta\sin\phi$

Consider $y = r\sin\theta\sin\phi$

$\frac{\partial y}{\partial r} = \sin\theta\sin\phi$; $\frac{\partial y}{\partial \theta} = r\cos\theta\sin\phi$; $\frac{\partial y}{\partial \theta} = r\sin\theta\cos\phi$

Consider $z = r\cos\theta$

$\frac{\partial z}{\partial r} = \cos\theta$; $\frac{\partial z}{\partial \theta} = -r\sin\theta$; $\frac{\partial z}{\partial \phi} = 0$

Sub in (1)

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\phi\cos\theta & r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$= \sin\theta\cos\phi[0 + r^2\sin^2\theta\cos\phi] - r\cos\phi\cos\theta[0 - r\sin\theta\cos\phi\cancel{\cos\theta}] - r\sin\theta\sin\phi[-r\sin^2\theta\sin\phi - r\sin\phi\cos^2\theta]$

$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r\sin\theta\sin^2\phi.$

$= \cos^2\phi\, r^2\sin\theta[\sin^2\theta + \cos^2\theta] + r^2\sin^2\phi\sin\theta$

$= r^2\sin\theta\cos^2\phi + r^2\sin^2\phi\sin\theta$

$= r^2\sin\theta$

(8) If $u = x + 3y + z^3$, $v = x^2yz$, $w = 2x^2 - xy$, evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at $(1,-1,0)$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Consider $u = x + 3y + z^3$

$\frac{\partial u}{\partial x} = 1$ $\quad \frac{\partial u}{\partial y} = 3$ $\quad \frac{\partial u}{\partial z} = 3z^2$

Consider $v = x^2yz$

$\frac{\partial v}{\partial x} = 2xyz$ $\quad \frac{\partial v}{\partial y} = x^2z(1)$ $\quad \frac{\partial v}{\partial z} = x^2y$

Consider $w = 2x^2 - xy$

$\frac{\partial w}{\partial x} = 4x - y$ $\quad \frac{\partial w}{\partial y} = 0 - x$ $\quad \frac{\partial w}{\partial z} = 0$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 3 & 3z^2 \\ 2xyz & x^2z & x^2y \\ 4x-y & -x & 0 \end{vmatrix}$$

$$= \begin{vmatrix} \overset{+}{1} & \overset{-}{3} & \overset{+}{0} \\ 0 & 0 & 0 \\ 5 & -1 & 0 \end{vmatrix}$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 1(0-1) - 3(0+5) + 0$$

$$= -1 - 15$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -16$$

(9) If $u = x + 3y^2$, $v = 4x^2yz$, $w = 2z^2 - xy$, evaluate Jacobian at $(1,-1,0)$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \rightarrow (1)$$

Consider $u = x + 3y^2$

$\frac{\partial u}{\partial x} = 1 + 0$ $\quad \frac{\partial u}{\partial y} = 6y$ $\quad \frac{\partial u}{\partial z} = 0$

Consider $V = 4x^2yz$

$\frac{\partial V}{\partial x} = 4yz(2x)$ $\frac{\partial V}{\partial y} = 4x^2z$ $\frac{\partial V}{\partial z} = 4x^2y$

Consider $W = 2z^2 - xy$

$\frac{\partial W}{\partial x} = -y$ $\frac{\partial W}{\partial y} = -x$ $\frac{\partial W}{\partial z} = 4z$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 6y & 0 \\ 4yz(2x) & 4x^2z & 4x^2y \\ -y & -x & 4z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 6(-1) & 0 \\ 0 & 0 & 4(1)^2(-1) \\ +1 & -1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -6 & 0 \\ 0 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix}$$

$= 1(0-4) + 6(0+4) + 0$

$= -4 + 24$

$= 20$

$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 20$

(10) $x+y+z=u$, $y+z=v$, $z=uvw$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Consider $x+y+z=u$ $y+z=v$ $z=uvw$

$x+v=u$ $y=v-z$

$x=u-v$ $y=u-uvw$

Consider $x=u-v$

$\frac{\partial x}{\partial u} = 1$ $\frac{\partial x}{\partial v} = -1$ $\frac{\partial x}{\partial w} = 0$

Consider $y = u - uvw$

$$\frac{\partial y}{\partial u} = -vw \qquad \frac{\partial y}{\partial v} = 1 - uw \qquad \frac{\partial y}{\partial w} = -uv$$

Consider $z = uvw$

$$\frac{\partial z}{\partial u} = vw \qquad \frac{\partial z}{\partial v} = uw \qquad \frac{\partial z}{\partial w} = uv$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1 & -1 & 0 \\ -vw & 1-uw & -uw \\ vw & uw & uv \end{vmatrix}$$

$= 1[(1-uw)uv - (-uv)(uw)] + 1[(-uv)(uv) - (-uv(vw))] + 0$

$= uv - u^2vw + u^2vw - uv^2w + uv^2w$

$= uv - \cancel{u^2vw} + \cancel{u^2vw} - \cancel{uv^2w} + \cancel{uv^2w}$

$= uv$

(11) If $u = 3x + 2y - z$, $v = x - 2y + z$, $w = x^2 + 2xy - xz$ then show that $J = 0$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Consider $u = 3x + 2y - z$

$$\frac{\partial u}{\partial x} = 3 \qquad \frac{\partial u}{\partial y} = 2 \qquad \frac{\partial u}{\partial z} = -1$$

Consider $v = x - 2y + z$

$$\frac{\partial v}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -2 \qquad \frac{\partial v}{\partial z} = 1$$

Consider $w = x^2 + 2xy - xz$

$$\frac{\partial w}{\partial x} = 2x + 2y - z \qquad \frac{\partial w}{\partial y} = 2x \qquad \frac{\partial w}{\partial z} = -x$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 3 & 2 & -1 \\ 1 & -2 & 1 \\ 2x+2y-z & 2x & -x \end{vmatrix}$$

$= 3(2x-2x) - 2[-x-(2x+2y-z)] - 1[2x-(-4x-4y+2x)]$

$= 3(0) - 2(-x-2x-2y+z) - (2x+4x+4y-2z)$

$= -2x + 4x + 4y - 2z + 2x + 4x + 4y - 2z$

$= -2x + 4x + 4y - 2z + 2x - 4x - 4y + 2z$

$= 0$.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

→ Total derivatives and chain rule.

Total derivative: If $u = f(x,y)$ then the total differential or the exact differential of u is defined as

$$du = \frac{\partial u}{\partial x} \cdot dx + \frac{\partial u}{\partial y} \cdot dy$$

differentiation of Composite functions

① If $u = f(x,y)$ where x and y are functions of independent variable t then u is called composite function of t and we can find $\frac{du}{dt}$ using

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}.$$

② If $u = f(x,y)$ where both x and y are function of two independent variable r, s and then u is called composite function of two variable r and s, we can find $\frac{\partial u}{\partial r}$ & $\frac{\partial u}{\partial s}$ using the formula

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial r}.$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Problems.

① If $u = x^3 + y^3$ where, $x = a\cos t$, $y = b\sin t$, find $\frac{du}{dt}$

$$u \to (x,y) \to t$$

$$\Rightarrow u \to t.$$

$\frac{du}{dt}$ exists.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= (3x^2)(-a\sin t) + (3y^2)(b\cos t)$$

$$= (3a^2\cos^2 t)(-a\sin t) + (3b^2\sin^2 t)(b\cos t)$$

$$= -3a^3\cos^2 t \cdot \sin t + 3b^3\sin^2 t \cdot \cos t$$

$$= 3\sin t\cos t\,(b^3\sin t - a^3\cos t)$$

② If $u = x^3y^2 + x^2y^3$, where $x = at^2$, $y = 2at$, find the total differential?

$$u \to (x,y) \to t.$$

$$u \to t.$$

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$

$$= (3x^2y^2 + 2xy^2)(2at) + (2x^3y + 3x^2y^2)(2a)$$

$$= [3(a^2t^4)(4a^2t^2) + 2(at^2)(2at)^3](2at) + [2(at^2)^3(2at) + 3(at^2)^2(2at)^2](2a)$$

$$\frac{du}{dt} = [12a^4t^6 + 2(at^2)(8a^3t^3)](2at) + [(4at)(a^3t^6) + 3(a^2t^4)(4a^2t^2)][2a]$$

$$= [12a^4t^6 + 16a^4t^5](2at) + [4a^4t^7 + 12a^4t^6](2a)$$

$$= 24a^5t^7 + 32a^5t^6 + 8a^5t^7 + 24a^5t^6$$

$$= 32a^5t^7 + 56a^5t^6$$

③ If $u = \tan^{-1}\left(\frac{y}{x}\right)$, where $x = e^t - e^{-t}$, $y = e^t + e^{-t}$, find $\frac{du}{dt}$

$u \to (x, y) \to t$

$u \to t$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \to ①$$

consider $u = \tan^{-1}(y/x)$

$$\frac{\partial u}{\partial x} = \frac{1}{1 + y^2/x^2} \times \frac{-y}{x^2}$$

$$= \frac{1}{\frac{x^2+y^2}{\cancel{x^2}}} \times \frac{-y}{\cancel{x^2}}$$

$$\frac{\partial u}{\partial x} = \frac{-y}{x^2+y^2}$$

$$= \frac{-(e^t + e^{-t})}{(e^t - e^{-t})^2 + (e^t + e^{-t})^2}$$

$$= \frac{-e^t - e^{-t}}{e^{2t} + e^{-2t} \cancel{- 2e^t e^{-t}} + e^{2t} + e^{-2t} \cancel{+ 2e^t e^{-t}}}$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{-e^t - e^{-t}}{2e^{2t} + 2e^{-2t}}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1 + y^2/x^2} \times \frac{1}{x}$$

$$= \frac{1}{\frac{x^2 + y^2}{x^2}} \times \frac{1}{\cancel{x}}$$

$$\frac{\partial u}{\partial y} = \frac{x}{x^2+y^2}$$

$$= \frac{e^t - e^{-t}}{(e^t - e^{-t})^2 + (e^t + e^{-t})^2}$$

$$\boxed{\frac{\partial u}{\partial y} = \frac{e^t - e^{-t}}{2e^{2t} + 2e^{-2t}}}$$

Sub in ①

$$\frac{du}{dt} = \frac{(-e^t - e^{-t})}{2e^{2t} + 2e^{-2t}} \times (e^t + e^{-t}) + \frac{(e^t - e^{-t})}{2e^{2t} + 2e^{-2t}} \times (e^t - e^{-t})$$

$$= \frac{-e^{2t} - e^{0} - e^{0} - e^{-2t} + e^{2t} - e^{0} - e^{0} + e^{-2t}}{2e^{2t} + 2e^{-2t}}$$

$$= \frac{-4}{2(e^{2t} + e^{-2t})}$$

$$\frac{du}{dt} = \frac{-2}{e^{2t} + e^{-2t}}$$

(4) If $u = x^2 + y^2 + z^2$, $x = e^{2t}$, $y = e^{2t}\cos 2t$, $z = e^{2t}\sin 2t$ find $\frac{du}{dt}$

$u \to (x, y, z) \to t$

$\frac{du}{dt}$ = exists.

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{du}{dt} = 2x \cdot e^{2t} \cdot 2 + 2y \cdot e^{2t} \cdot 2 \cdot \cos 2t + e^{2t}(-\sin 2t \; 2) + e^{2t} \cdot 2 \sin 2t + e^{2t} \cdot \cos 2t \cdot 2.$$

$$\frac{du}{dt} = 4x \cdot e^{2t} + 4y \cdot e^{2t} \cdot \cos 2t \cancel{- 2e^{2t} \sin 2t} \cancel{+ 2e^{2t} \cdot \sin 2t} + 2e^{2t} \cdot \cos 2t$$

$$\frac{du}{dt} = 4x \cdot e^{2t} + 4y \cdot e^{2t} \cdot \cos 2t + 2e^{2t} \cos 2t$$

$$\frac{du}{dt} = 4\left[(e^{2t}) \cdot e^{2t}\right] + 4\left[(e^{2t} \cdot \cos 2t) e^{2t} \cdot \cos 2t + 2e^{2t} \cos 2t\right.$$

$$\frac{du}{dt} = 4e^{4t} + 4e^{4t} \cdot \cos 2t + 2e^{2t} \cos 2t$$

$$\frac{du}{dt} = 2(2e^{4t} + 2e^{4t} \cdot \cos^2 2t + e^{2t} \cos 2t)$$

(5) If $u = \log(x + y + z)$, $x = e^{-t}$, $y = \sin t$, $z = \cos t$, find $\frac{du}{dt}$

$u \to (x, y, z) \to t$

$\frac{du}{dt}$ = exists

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{du}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \frac{dz}{dt}$$

$$= \frac{1}{x+y+z} e^{-t}(-1) + \frac{1}{x+y+z} \cos t + \frac{1}{x+y+z} -\sin t$$

$$\frac{du}{dt} = \frac{1}{x+y+z} [-e^{-t} + \cos t - \sin t]$$

* (6) If $u = f(x-y, y-z, z-x)$ S.T $u_x + u_y + u_z = 0$

Let $\boxed{p = x-y}$, $\boxed{q = y-z}$, $\boxed{r = z-x}$.

$\therefore u \to f(p, q, r) \to f(x, y, z)$

$u \to (x, y, z)$

$\therefore \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ exists.

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$u_x = \frac{\partial u}{\partial p} \cdot 1 + \frac{\partial u}{\partial q} 0 + \frac{\partial u}{\partial r} (-1)$$

$$\boxed{u_x = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}} \rightarrow (1)$$

Now, $\frac{\partial u}{\partial y} = u_y = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y}$

$$u_y = \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q} \cdot 1 + \frac{\partial u}{\partial r}(0)$$

$$\boxed{u_y = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = u_z = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z}$$

$$u_z = \frac{\partial u}{\partial p} \cdot 0 + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial r} \cdot 1$$

$$\boxed{u_z = -\frac{\partial u}{\partial z} + \frac{\partial u}{\partial r}} \rightarrow (3)$$

Adding (1), (2), (3)

$$u_x + u_y + u_z = 0$$

(7) If $u = f(2x-3y,\ 3y-4z,\ 4z-2x)$ then find the value of $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z}$ (or) $6u_x + 4u_y + 3u_z$

Let $\boxed{p = 2x-3y}$ $\boxed{q = 3y-4z}$, $\boxed{r = 4z-2x}$

$$u \rightarrow f(p,q,r) \rightarrow f(x,y,z)$$

$$u \rightarrow (x,y,z)$$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ exists.

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} + \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x}$$

$$u_x = \frac{\partial u}{\partial p} \cdot 2 + \frac{\partial u}{\partial q} \cdot 0 + \frac{\partial u}{\partial r} \cdot (-2)$$

xly by $\frac{1}{2}$

$$\boxed{\frac{1}{2} u_x = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r}} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = U_y = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$U_y = \frac{\partial u}{\partial p}\cdot(-3) + \frac{\partial u}{\partial q}\cdot 3 + \frac{\partial u}{\partial r}(0)$$

$\times$ly $1/3$

$$\boxed{\frac{1}{3}U_y = -\frac{\partial u}{\partial p} + \frac{\partial u}{\partial q}} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = U_z = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$U_z = \frac{\partial u}{\partial p}\,0 + \frac{\partial u}{\partial q}(-4) + \frac{\partial u}{\partial r}\cdot 4$$

$\times$ly by $1/4$

$$\boxed{\frac{1}{4}U_z = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r}} \rightarrow (3)$$

Add (1), (2), (3)

$$\frac{1}{2}U_x + \frac{1}{3}U_y + \frac{1}{4}U_z = 0$$

** (8) If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$, then P.T $xU_x + yU_y + zU_z = 0$

Let $\boxed{p = \frac{x}{y}}$ $\boxed{q = \frac{y}{z}}$ $\boxed{r = \frac{z}{x}}$

$u \rightarrow f(p, q, r) \rightarrow f(x, y, z)$

$u \rightarrow (x, y, z)$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ exists.

$$\frac{\partial u}{\partial x} = U_x = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$U_x = \frac{\partial u}{\partial p}\cdot\frac{1}{y} + \frac{\partial u}{\partial q}\cdot 0 + \frac{\partial u}{\partial r}\cdot\left(\frac{-z}{x^2}\right)$$

$\times$ly by x

$$xU_x = \frac{\partial u}{\partial p}\cdot\frac{x}{y} - \frac{\partial u}{\partial r}\frac{z}{x} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = u_y = \frac{\partial u}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial y}$$

$$u_y = \frac{\partial u}{\partial p} \cdot \frac{-x}{y^2} + \frac{\partial u}{\partial q} \cdot \frac{1}{z} + \frac{\partial u}{\partial r} \cdot 0$$

xly by y

$$\boxed{y u_y = \frac{-x}{y}\frac{\partial u}{\partial p} + \frac{y}{z}\frac{\partial u}{\partial q}} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = u_z = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

$$u_z = \frac{\partial u}{\partial p} \cdot 0 + \frac{\partial u}{\partial q} \cdot \frac{-y}{z^2} + \frac{\partial u}{\partial r} \cdot \frac{1}{x}$$

xly by z

$$\boxed{z u_z = \frac{-y}{z}\frac{\partial u}{\partial q} + \frac{\partial u}{\partial r} \cdot \frac{z}{x}} \rightarrow (3)$$

Adding (1), (2) & (3)

$$x u_x + y u_y + z u_z = 0$$

(9) If $u = f\left(xz, \frac{y}{z}\right)$ P.T $x u_x - y u_y - z u_z = 0$

Let $\boxed{p = xz}$, $\boxed{q = \frac{y}{z}}$

$u \rightarrow f(p, q) \rightarrow (x, y, z)$

$u \rightarrow (x, y, z)$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ exists.

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x}$$

$$u_x = \frac{\partial u}{\partial p} \cdot z + \frac{\partial u}{\partial q} \cdot 0$$

xly by x

$$\boxed{x u_x = xz\frac{\partial u}{\partial p}} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = u_y = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y}$$

$$u_y = \frac{\partial u}{\partial p}\cdot 0 + \frac{\partial u}{\partial q}\cdot\frac{1}{z}$$

x^ly by $(-y)$

$$\boxed{-y\,u_y = \frac{-y}{z}\frac{\partial u}{\partial q}} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = u_z = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z}.$$

$$u_z = \frac{\partial u}{\partial p}\cdot x + \frac{\partial u}{\partial q}\cdot\frac{-y}{z^2}$$

x^ly by $(-z)$

$$\boxed{-z\,u_z = -xz\frac{\partial u}{\partial p} + \frac{y}{z}\frac{\partial u}{\partial q}} \rightarrow (3)$$

Adding (1), (2) & (3)

$$x u_x - y u_y - z u_z = 0$$

*(10) If $u = f(y-z,\ z-x,\ x-y)$ then PT $u_x + u_y + u_z = 0$

Let $p = y-z$ $\quad q = z-x$ $\quad r = x-y$

$u \rightarrow f(p, q, r) \rightarrow (x, y, z)$

$u \rightarrow (x, y, z)$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial x}$$

$$= \frac{\partial u}{\partial p}\cdot(1) + \frac{\partial u}{\partial q}(0) + \frac{\partial u}{\partial r}(-1)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} - \frac{\partial u}{\partial r} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = u_y = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial y} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial y}$$

$$= \frac{\partial u}{\partial p}(-1) + \frac{\partial u}{\partial q}(1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = U_z = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\cdot\frac{\partial r}{\partial z}$$

$$= \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}(-1) + \frac{\partial u}{\partial c}(1)$$

$$\frac{\partial u}{\partial z} = -\frac{\partial u}{\partial q} + \frac{\partial u}{\partial c} \rightarrow (3)$$

Adding (1), (2), (3)

$$u_x + u_y + u_z = 0$$

(11) If $u = u\left(\frac{y-x}{xy}, \frac{z-x}{zx}\right)$ find $x^2 U_x + y^2 U_y + z^2 U_x$

Let $p = \frac{y-x}{xy}$ $\quad q = \frac{z-x}{zx}$

$u \rightarrow f(p, q) \rightarrow (x, y, z)$

$u \rightarrow x, y, z$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ exists

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x}$$

$$= \frac{\partial u}{\partial p}\cdot\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial q}\cdot\left(\frac{1}{x^2}\right)$$

$$x^2\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} \rightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\cdot\frac{\partial p}{\partial y} + \frac{\partial u}{\partial q}\cdot\frac{\partial q}{\partial x}$$

$$= \frac{1}{y^2}\frac{\partial u}{\partial p} + 0$$

$$\frac{\partial u}{\partial y} = \frac{1}{y^2}\frac{\partial u}{\partial p}$$

x ly by y^2

$$y^2\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \rightarrow (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z}$$

$$= \frac{\partial u}{\partial p}\cdot(0) + \frac{\partial u}{\partial q}\left(\frac{1}{z^2}\right)$$

$$\frac{\partial u}{\partial z} = \frac{1}{z^2}\frac{\partial u}{\partial q}$$

x/y by z^2

$$z^2\frac{\partial u}{\partial z} = \frac{\partial u}{\partial q} \rightarrow (3)$$

Adding (1), (2) & (3)

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0.$$

→ Maxima and Minima for a function of 2 variables

A function $f(x,y)$ is said to have an extreme value at a point (a,b) if $f(x,y)$ is either maximum or minimum.

Note: The necessary Conditions for a function $f(x,y)$ to have either a maximum or a minimum at a point (a,b) are $f_x(a,b)=0$, $f_y(a,b)=0$

The points (a,b) where a and b satisfy $f_x(a,b)=0$ $f_y(a,b)=0$ are Called the Stationary points or Critical points of the function $f(x,y)$.

Working Rule.

Step 1: Find the stationary points (a,b) such that $f_x=0$ and $f_y=0$

Step 2: Find the 2^{nd} order Partial derivatives

f_{xx}, f_{xy}, f_{yy} denoting $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$
Evaluate A, B, C at the stationary points and compute the corresponding value of $AC - B^2$

Step 3: (i) If $AC - B^2$ is > 0 and $A < 0$ at the point (a, b) then f has maximum at (a, b) and $f(a, b)$ is the maximum value.

(ii) If $AC - B^2$ is > 0 and $A > 0$ at the point (a, b) then f has minimum at (a, b) and $f(a, b)$ is the minimum value.

Note :- If $AC - B^2 < 0$ or $AC - B^2 = 0$ or $A = 0$ then the point (a, b) is called the "Saddle point" in this cases

Problems

1) Find the extreme values of $f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$

$$f(x, y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x \rightarrow (1)$$

$$f_x = 3x^2 + 3y^2 - 30x - 0 + 72$$

$$f_y = 0 + 3x \cdot 2y - 0 - 30y + 0$$

$$f_y = 6xy - 30y$$

To find critical points, $f_x = 0$, $f_y = 0$

$3x^2 + 3y^2 - 30x + 72 = 0 \rightarrow (2)$; $6xy - 30y = 0$

$\Rightarrow 6y(x - 5) = 0$

$y = 0 \qquad x - 5 = 0$

$x = 5$

Put $x = 5$ in (2)

$$75 + 3y^2 - 150 + 72 = 0$$

$$3y^2 - 3 = 0 \qquad \div 3$$

$$y^2 - 1 = 0$$

$$y^2 = 1$$

$$y = \pm 1$$

Put $y=0$ in ⑤

$3x^2 - 30x + 72 = 0 \quad \div 3.$

$x^2 - 10x + 24 = 0$

$x = 6, 4$

$$24 \to -6, -4$$

Critical points are

$(5,1)\ (5,-1)\ (6,0)\ (4,0)$

Next, $A = f_{xx} = 6x - 30$

$B = f_{xy} = 6y$

$C = f_{yy} = 6x - 30.$

	$(5,1)$	$(5,-1)$	$(6,0)$	$(4,0)$
$A = 6x-30$	0	0	$6>0$	$-6<0$
$B = 6y$	6	-6	0	0
$C = 6x-30$	0	0	6	-6
$AC - B^2$	$-36<0$	$-36<0$	$36>0$	$36>0$
Conclusion	Saddle point	Saddle poin	Min point	Max point.

$\therefore$ Min value $= f(6,0)$

$= 108$

Max value $= f(4,0)$

$= 112.$

② Examine the function $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ for its extreme value.

$f(x,y) = 2 + 2x + 2y - x^2 - y^2 \to ①.$

$f_x = 2 - 2x;$

$f_y = 2 - 2y$

To find the critical points $f_x = 0, \quad f_y = 0$

$2 - 2x = 0 \quad ; \quad 2 - 2y = 0$

$\div 2. \quad 1 - x = 0 \qquad 1 - y = 0$

$x = 1 \qquad y = 1$

stationary point is $(1,1)$

$A = f_{xx} = -2$

$B = f_{xy} = 0$

$C = f_{yy} = -2$

$\therefore AC - B^2 = 4 - 0 = 4 > 0$

and $A = -2 < 0$

$\therefore (1,1)$ is a max point.

Max value $= f(1,1)$

$= 2+2+2-1-1$

$= 4$.

③ Find the extreme values of $f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f(x,y) = x^3 + y^3 - 3x - 12y + 20 \rightarrow ①$

$f_x = 3x^2 - 3$, $f_y = 3y^2 - 12$

To find critical points

$f_x = 0$, $f_y = 0$

$3x^2 - 3 = 0$, $3y^2 - 12 = 0$

$\div 3$ $x^2 - 1 = 0$, $y^2 - 4 = 0$.

$x^2 = 1$, $y^2 = 4$

$\boxed{x = \pm 1}$, $\boxed{y = \pm 2}$

Stationary points are $(1,2)$ $(1,-2)$, $(-1,2)$ $(-1,-2)$

$A = f_{xx} = 6x$

$B = f_{xy} = 0$

$C = f_{yy} = 6y$

	$(1,2)$	$(1,-2)$	$(-1,2)$	$(-1,-2)$
$A = 6x$	$6 > 0$	6	-6	$-6 < 0$
$B = 0$	0	0	0	0
$C = 6y$	12	-12	12	-12
$AC - B^2$	$72 > 0$	$-72 < 0$	$-72 < 0$	$72 > 0$
Conclusion	min point	Saddle point	Saddle point	Max point

Min value $= f(1,2)$
$= 2$

Max value $= f(-1,-2)$
$= 38$

④ S.T the function $f(x,y) = x^3 + y^3 - 63x - 63y + 12xy$ is max at $(-7,-7)$ Hence find the max value

$$f(x,y) = x^3 + y^3 - 63x - 63y + 12xy$$

$f_x = 3x^2 - 63 + 12y$; $f_y = 3y^2 - 63 + 12x$.

Let $A = f_{xx} = 6x$
$B = f_{xy} = 12$
$C = f_{yy} = 6y$

at $(-7,-7)$; $A = -42$
$B = 12$
$C = -42$

$AC - B^2 = (-42 \times -42) - 12^2$
$= 1764 - 144$
$= 1620 > 0$

and $A = -42 < 0$.
$\therefore (-7,-7)$ is a max point

Max value $= f(-7,-7) = 784$.

⑤ S.T the function $f(x,y) = xy(a-x-y)$ is max at the point $\left(\frac{a}{3}, \frac{a}{3}\right)$ Hence find the max value.

$$f(x,y) = axy - x^2y - xy^2$$

$f_x = ay - 2xy - y^2$; $f_y = ax - x^2 - 2xy$

$A = f_{xx} = -2y$
$B = f_{xy} = a - 2x - 2y$
$C = f_{yy} = -2x$

at $\left(\frac{a}{3}, \frac{a}{3}\right)$; $A = -\frac{2a}{3}$

$B = a - \frac{2a}{3} - \frac{2a}{3} = \frac{3a-2a-2a}{3}$

$B = -\frac{a}{3}$

$C = -\frac{2a}{3}$

$\therefore AC - B^2 = \left(-\frac{2a}{3} \times -\frac{2a}{3}\right) - \left(\frac{-a}{3}\right)^2$

$AC - B^2 = \frac{4a^2}{9} - \frac{a^2}{9} = \frac{3a^2}{9} = \frac{a^2}{3} > 0$

& $A = -\frac{2a}{3} < 0$ $(\because a > 0)$

$\therefore \left(\frac{a}{3}, \frac{a}{3}\right)$ is a max point

$\therefore$ Max value

$= f\left(\frac{a}{3}, \frac{a}{3}\right)$

$= \frac{a}{3} \cdot \frac{a}{3}\left[a - \frac{a}{3} - \frac{a}{3}\right]$

$= \frac{a^2}{9}\left[\frac{3a-a-a}{3}\right]$

$= \frac{a^2}{9} \times \frac{a}{3}$

Max value $= \frac{a^3}{27}$