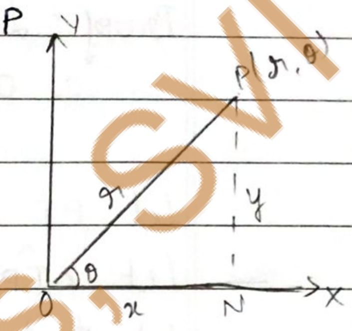
MODULE 1 : Calculus

→ Polar Curves $r = f(\theta)$



Let P be a Point on XY Plane join OP

Let $OP = r$ and $X\hat{O}P = \theta$

$\therefore P(r, \theta) = P = (r, \theta)$

Here 'O' (origin) is called the Pole

'OX' is called initial line.

$OP = r$ is called radius vector

$X\hat{O}P = \theta$ is called Polar angle

WKT, from fig.

$$\sin\theta = \frac{opp}{hyp} = \frac{y}{r} \qquad y = r\sin\theta$$

$$\cos\theta = \frac{adj}{hyp} = \frac{x}{r}. \qquad x = r\cos\theta.$$

Squaring and adding

$$y^2 + x^2 = r^2\sin^2\theta + r^2\cos^2\theta.$$

$$y^2 + x^2 = r^2(\sin^2\theta + \cos^2\theta)$$

$$y^2 + x^2 = r^2$$

$$r = \sqrt{x^2+y^2}$$

$$\therefore \frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

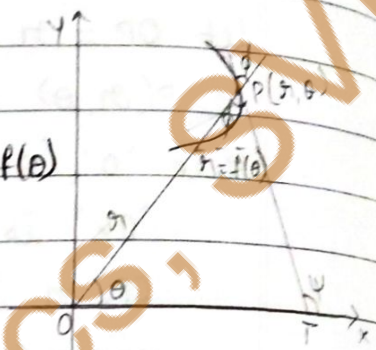
$\therefore r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ represents the relation between Polar and Cartesian co-ordinates

→ Angle Between Radius vector and Tangent.

Prove that: $\tan\phi = r\dfrac{d\theta}{dr}$

Proof: Let $P(r,\theta)$ be a Point on the Curve $r = f(\theta)$

$\therefore OP = r$ and $X\hat{O}P = \theta$



Let PT be the tangent to the curve $r = f(\theta)$

Let $X\hat{T}P = \psi$

Let ϕ be the angle b/w the radius vector OP and the tangent PT i.e $O\hat{P}T = \phi$

From fig,

$\psi = \phi + \theta$ [$\because$ Exterior angle = Sum of the Interior angles]

$\tan\psi = \tan(\phi + \theta)$

$$\tan\psi = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta} \longrightarrow (1)$$

$\tan(a+b) = \dfrac{\tan a + \tan b}{1 - \tan a \tan b}$

If $P(x,y)$ are Cartesian Co-ordinates of P then we have,

$x = r\cos\theta$, $y = r\sin\theta$.

Also, WKT $\tan\psi = \dfrac{dy}{dx}$ = Slope of the tangent

$$\tan\psi = \frac{dy/d\theta}{dx/d\theta}$$

$$\tan\psi = \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)}$$

(Product rule)

$$\tan\psi = \frac{r.\cos\theta + \sin\theta.\frac{dr}{d\theta}}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$$

Let $\dfrac{dr}{d\theta} = r'$

$$\tan\psi = \frac{r\cos\theta + r'\sin\theta}{r'\cos\theta - r\sin\theta}$$

$\div$ Both num & deno by $r'\cos\theta$.

$$\tan\psi = \frac{\frac{r\cos\theta}{r'\cos\theta} + \frac{r'\sin\theta}{r'\cos\theta}}{\frac{r'\cos\theta}{r'\cos\theta} - \frac{r\sin\theta}{r'\cos\theta}}$$

$$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta} \rightarrow (2)$$

Comparing (1) & (2).

$$\tan\phi = \frac{r}{r'} = \frac{r}{dr/d\theta}$$

$$\boxed{\tan\phi = r\frac{d\theta}{dr}}$$

This is the expression for angle b/w radius vector and tangent to the curve $r = f(\theta)$

Note : $\boxed{\cot\phi = \frac{1}{r}\frac{dr}{d\theta}}$

Important Formulae

1. $\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$, $\cot\left(\frac{\pi}{2} - \theta\right) = \tan\theta$

2. $\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$, $\cot\left(\frac{\pi}{2} + \theta\right) = -\tan\theta$

3. $\tan\left(\frac{\pi}{4} + \theta\right) = \frac{\tan\pi/4 + \tan\theta}{1 - \tan\pi/4\,\tan\theta}$, $\cancel{\cot\left(\frac{\pi}{4} + \theta\right)} = \frac{1 + \tan\theta}{1 - \tan\theta}$

$\cancel{1 + \cos\theta} = \frac{\cancel{1 - \tan\theta}}{1 + \tan\theta}$

4. $\cot\left(\frac{\pi}{4} + \theta\right) = \frac{1 - \tan\theta}{1 + \tan\theta}$

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5. $1+\cos\theta = 2\cos^2\theta/2$

$1-\cos\theta = 2\sin^2\theta/2$

I Find the Rad Angle between the Radius vector and the tangent for the following Curves.

1. $\frac{2a}{r} = 1-\cos\theta$ at $\theta = \frac{2\pi}{3}$

Solu: $\frac{2a}{r} = 1-\cos\theta$

taking log on both sides

$\log_e\left(\frac{2a}{r}\right) = \log_e(1-\cos\theta)$

$\log(2a) - \log(r) = \log(1-\cos\theta)$

diff. wrt θ

$0 - \frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{1}{1-\cos\theta} \times (0+\sin\theta)$

$\Rightarrow \frac{-1}{r}\cdot\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$

$\cot\phi = \frac{-\sin\theta}{1-\cos\theta}$

$= \frac{-\cancel{2}\sin\theta/2\cdot\cos\theta/2}{\cancel{2}\sin^2\theta/2}$

$= \frac{-\cos\theta/2}{\sin\theta/2}$

$\cot\phi = -\cot\frac{\theta}{2}$

$\cot\phi = \cot\left(-\frac{\theta}{2}\right)$

$\boxed{\phi = \frac{-\theta}{2}}$

at $\theta = \frac{2\pi}{3}$, $\phi = -\theta/2$

$\boxed{\phi = \pi/3}$

2. $r^2 = a^2 \sin 2\theta$

Solu: $r^2 = a^2 \sin 2\theta$

taking log on both sides

$\log(r^2) = \log(a^2 \sin 2\theta)$ $\qquad \log(m^n) = n \log m$

$2 \log r = \log a^2 + \log \sin(2\theta)$

diff w.r.t. θ

$2 \cdot \frac{1}{r} \frac{dr}{d\theta} = 0 + \frac{1}{\sin 2\theta} \times \cos 2\theta \times 2.$ $\qquad \frac{1}{r} \cdot \frac{dr}{d\theta} = \cot\phi$

$\cancel{2} \cot\phi = \cancel{2} \cot 2\theta.$

$\cot\phi = \cot 2\theta$

$\boxed{\phi = 2\theta}$

3. $r = a(1+\cos\theta)$ at $\theta = \pi/3$

Solu: $r = a(1+\cos\theta)$

taking log on both sides

$\log r = \log[a(1+\cos\theta)]$ $\qquad \log a \cdot b = \log a + \log b$

$\log r = \log a + \log(1+\cos\theta)$

diff w.r.t θ.

$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{1+\cos\theta} \times -\sin\theta$

$\cot\phi = \frac{-\sin\theta}{1+\cos\theta}$

$\cot\phi = \frac{-\cancel{2} \sin\theta/2 \cdot \cos\theta/2}{\cancel{2} \cos^2\theta/2}$

$\cot\phi = -\tan\theta/2$

$\cot\phi = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$

$\phi = \frac{\pi}{2} + \frac{\theta}{2}$

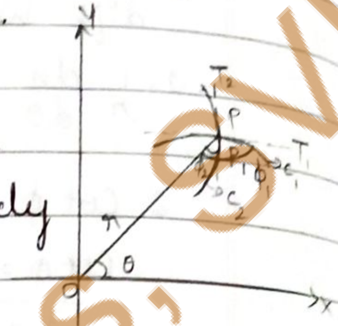
at $\theta = \pi/3$

$\phi = \frac{\pi}{2} + \frac{\pi}{6}$

$= \frac{3\pi + \pi}{6} = \frac{4\pi}{6} \Rightarrow \boxed{\phi = \frac{2\pi}{3}}$

Angle between two Polar Curves

Suppose two curves C_1 and C_2 intersect at the Point P then, the angle of intersection of the curves C_1 and C_2 at the Point P is the angle between the tangents PT_1 and PT_2 to C_1 and C_2 respectively.



Suppose the radius vector OP makes the angles ϕ_1 and ϕ_2 with PT_1 and PT_2 respectively then the difference $|\phi_1 - \phi_2|$ is the angle between PT_1 and PT_2 this angle is determined by using the formula.

$$\tan|\phi_1 - \phi_2| = |\tan(\phi_1 - \phi_2)| = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$$

<u>Note</u>: If $|\phi_1 - \phi_2| = \pi/2$ or $\tan\phi_1 \cdot \tan\phi_2 = -1$ then the curves are said to be intersect orthogonally

$$\frac{1}{\cot\phi_1} \cdot \frac{1}{\cot\phi_2} = -1$$

$$\cot\phi_1 \ \cot\phi_2 = -1$$

1. Find the Angle between two Curves $r = a(1+\sin\theta)$ and $r = b(1-\sin\theta)$

<u>Sol</u>: $r = a(1+\sin\theta)$

take log on b.s

$\log r = \log[a(1+\sin\theta)]$

$\frac{1}{r} \cdot \frac{dr}{d\theta} = \log a + \log(1+\sin\theta)$

D.w.r.t. θ

$$\cot\phi = 0 + \frac{1}{1+\sin\theta} \times \cos\theta$$

$$\cot\phi_1 = \frac{\cos\theta}{1+\sin\theta}$$

$$\tan\phi_1 = \frac{1+\sin\theta}{\cos\theta}$$

$r = b(1-\sin\theta)$

$\log r = \log[b(1-\sin\theta)]$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = \log b + \log(1-\sin\theta)$

$\cot\phi = \frac{1}{1-\sin\theta} \times (-\cos\theta)$

$\cot\phi_2 = \frac{-\cos\theta}{1-\sin\theta}$

$\tan\phi_2 = \frac{1-\sin\theta}{-\cos\theta}$

$\tan\phi_1 \cdot \tan\phi_2 = \frac{1+\sin\theta}{\cos\theta} \cdot \frac{1-\sin\theta}{-\cos\theta}$

$= \frac{1-\sin^2\theta}{-\cos^2\theta}$

$= \frac{\cos^2\theta}{1-\cos^2\theta}$

$\tan\phi_1 \cdot \tan\phi_2 = -1$. $\therefore$ ~~Does~~ The curves intersect orthogonally

2. $r = 4\sec^2(\theta/2)$ and $r = 9\operatorname{cosec}^2(\theta/2)$

Solu:- $r = 4\sec^2(\theta/2)$

take log on b.s

$\log r = \log[4\sec^2(\theta/2)]$

$\frac{1}{r}\frac{dr}{d\theta} = \log 4 + \log\sec^2(\theta/2)$

D.W.r.t θ

$\cot\phi = \log 4 + 2\log\sec(\theta/2)$

$= 0 + 2\cdot\frac{1}{\sec(\theta/2)} \sec(\theta/2)\cdot\tan(\theta/2)\cdot\frac{1}{2}$

$\cot\phi = \tan(\theta/2)$

$\cot\phi_1 = \cot\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

$\phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$

$r = 9\,\mathrm{cosec}^2(\theta/2)$

take log on b.s

$\log r = \log[9(\mathrm{cosec}^2(\theta/2))]$

$= \log 9 + \log \mathrm{cosec}^2(\theta/2)$

$\log r = \log 9 + 2\log \mathrm{cosec}(\theta/2)$

D.w.r.t. θ.

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \cancel{2}\cdot\frac{1}{\mathrm{cosec}(\theta/2)}\cdot -\mathrm{cosec}(\theta/2)\cot(\theta/2)\cdot\frac{1}{\cancel{2}}$$

$\cot\phi_2 = -\cot(\theta/2)$

$\phi_2 = -\frac{\theta}{2}$

$$|\phi_1 - \phi_2| = \left|\frac{\pi}{2} - \frac{\theta}{2} + \frac{\theta}{2}\right|$$

$$= \frac{\pi}{2}$$

$\therefore$ The curves intersect orthogonally.

3. $r^2\sin 2\theta = a^2$ and $r^2\cos 2\theta = b^2$

take log on b.s.

$\log[r^2\sin 2\theta] = \log a^2$

$\log r^2 + \log\sin 2\theta = \log a^2$.

$2\log r + \log\sin 2\theta = 2\log a$.

D.w.r.t θ.

$$\frac{2}{r}\cdot\frac{dr}{d\theta} + \frac{1}{\sin 2\theta}\cos 2\theta\cdot 2 = 0$$

$$\frac{\cancel{2}}{r}\frac{dr}{d\theta} = -\cot 2\theta\cdot\cancel{2}$$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = -\cot 2\theta$$

$\cot\phi_1 = -\cot 2\theta$

$\phi_1 = -2\theta$

$r^2 \cos 2\theta = b^2$

take log on b.s

$\log [r^2 \cos 2\theta] = \log b^2$

$\log r^2 + \log \cos 2\theta = \log b^2$

$2 \log r + \log \cos 2\theta = 2 \log b$.

D.w.r.t θ.

$\frac{2}{r} \cdot \frac{dr}{d\theta} + \frac{1}{\cos 2\theta} \cdot (-\sin 2\theta) \cdot 2 = 0$

$2 \cot \phi_2 - 2 \tan 2\theta = 0$.

$2 \cot \phi_2 = 2 \tan 2\theta$

$\cot \phi_2 = \tan 2\theta$.

$\therefore \cot \phi_1 \cdot \cot \phi_2 = -\cot 2\theta \times \tan 2\theta$

$= -\frac{1}{\cancel{\tan 2\theta}} \times \cancel{\tan 2\theta}$

$= -1$

$\therefore$ The Curves intersect orthogonally.

4. $r = 2\sin\theta$ and $r = 2(\sin\theta + \cos\theta)$

$r = 2\sin\theta$

take log on b.s

$\log r = \log (2\sin\theta)$

$\log r = \log 2 + \log \sin\theta$

diff w.r.t θ.

$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\sin\theta} \times \cos\theta$.

$\cot \phi_1 = \cot\theta$.

$\boxed{\phi_1 = \theta}$

$r = 2(\sin\theta + \cos\theta)$

take log on b.s

$\log r = \log [2(\sin\theta + \cos\theta)]$

$\log r = \log 2 + \log(\sin\theta + \cos\theta)$

diff w.r.t θ.

$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\sin\theta + \cos\theta} \times \cos\theta - \sin\theta$

$\cot \phi_2 = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$

$= \frac{\cos\theta \left(1 - \frac{\sin\theta}{\cos\theta}\right)}{\cos\theta \left(1 + \frac{\sin\theta}{\cos\theta}\right)}$

$= \frac{1 - \tan\theta}{1 + \tan\theta}$

$\cot \phi_2 = \cot\left(\frac{\pi}{4} + \theta\right)$

$\boxed{\phi_2 = \frac{\pi}{4} + \theta}$

$$|\phi_1 - \phi_2| = \left|\theta - \left(\frac{\pi}{4} + \theta\right)\right|$$

$$= \left|\theta - \frac{\pi}{4} - \theta\right|$$

$$= \left|-\frac{\pi}{4}\right|$$

$$\boxed{|\phi_1 - \phi_2| = \frac{\pi}{4}}$$

5. $r = a(1+\cos\theta)$ & $r = b(1-\cos\theta)$

soln: $r = a(1+\cos\theta)$

take log on b.s

$\log r = \log a(1+\cos\theta)$

$\log r = \log a + \log(1+\cos\theta)$

d.w.r.t θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{1+\cos\theta} \times -\sin\theta$$

$$\cot\phi_1 = \frac{-\sin\theta}{1+\cos\theta}$$

$r = b(1-\cos\theta)$

take log on b.s

$\log r = \log b(1-\cos\theta)$

$\log r = \log b + \log(1-\cos\theta)$

d.w.r.t θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta} \times \sin\theta$$

$$\cot\phi_2 = \frac{\sin\theta}{1-\cos\theta}$$

$$\cot\phi_1 \cdot \cot\phi_2 = \frac{-\sin\theta}{1+\cos\theta} \cdot \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{-\sin^2\theta}{1-\cos^2\theta}$$

$$= \frac{-\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}}$$

$$= -1$$

$\therefore$ The curves intersect orthogonally

6. $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$.

$r^n = a^n \cos n\theta$

take log on b.s

$\log r^n = \log(a^n \cos n\theta)$

$n\frac{1}{r}\cdot\frac{dr}{d\theta} = \log a^n + \log \cos n\theta$

$r^n = b^n \sin n\theta$

take log on b.s

$\log r^n = \log(b^n \sin n\theta)$

$\log r^n = \log b^n + \log \sin n\theta$

diff w.r.t θ

$n \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos n\theta} \times -\sin n\theta \times n$

$\cancel{n} \cot\phi_1 = -\cancel{n} \tan n\theta$

$\cot\phi_1 = -\tan n\theta$

diff w.r.t θ.

$n \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin n\theta} \times \cos n\theta \times n$

$\cancel{n} \cot\phi_2 = \cancel{n} \cot n\theta$

$\cot\phi_2 = \cot n\theta$

$\cot\phi_1 \cdot \cot\phi_2 = -\tan n\theta \times \frac{1}{\tan n\theta}$

$= -1$.

$\therefore$ The curves intersect orthogonally.

7. $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

$r^2 \sin 2\theta = 4$

take log on b.s

$\log(r^2 \sin 2\theta) = \log 4$

$2\log r + \log \sin 2\theta = \log 4$

diff. w.r.t θ.

$2 \cdot \frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\sin 2\theta} \times \cos 2\theta \cdot 2 = 0$

$2\cot\phi_1 + 2\cot 2\theta = 0$.

$\cot\phi_1 = -\cot 2\theta$

$\phi_1 = -2\theta$

$r^2 = 16 \sin 2\theta$

take log on b.s

$\log r^2 = \log(16 \sin 2\theta)$

$2\log r = \log 16 + \log \sin 2\theta$.

diff w.r.t θ.

$2 \cdot \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin 2\theta} \times \cos 2\theta \cdot 2$.

$\cancel{2}\cot\phi_2 = \cancel{2}\cot 2\theta$

$\cot\phi_2 = \cot 2\theta$.

$\phi_2 = 2\theta$.

$|\phi_1 - \phi_2| = |-2\theta - 2\theta|$

$= |-4\theta|$

$|\phi_1 - \phi_2| = 4\theta \rightarrow (1)$

To find θ

Consider $r^2 \sin 2\theta = 4$; $r^2 = 16 \sin 2\theta$

$r^2 = \frac{4}{\sin 2\theta}$

$\frac{\cancel{4}\,1}{\sin 2\theta} = \cancel{16}^{4} \sin 2\theta$

$1 = 4 \sin^2 2\theta$.

$\sin^2 2\theta = 1/4$

$\sin 2\theta = 1/2$

$2\theta = \sin^{-1}(1/2)$

$2\theta = \pi/6$

$\theta = \pi/12$

Subtitute in eqⁿ (1)

$|\phi_1 - \phi_2| = 4 \times \frac{\pi}{12}$

$= \frac{\pi}{3}$

5. $r = a(1+\sin\theta)$ & $r = a(1-\cos\theta)$

$r = a(1+\sin\theta)$

take log on b.s

$\log r = \log[a(1+\sin\theta)]$

$\log r = \log a + \log(1+\sin\theta)$

diff w.r.t θ

$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{1+\sin\theta} \times \cos\theta$

$\cot\phi_1 = \frac{\cos\theta}{1+\sin\theta}$

$= \frac{\cos^2\theta/2 - \sin^2\theta/2}{\cos^2\theta/2 + \sin^2\theta/2 + 2\sin\theta/2 \cdot \cos\theta/2}$

$= \frac{\cancel{(\cos\theta/2 + \sin\theta/2)}(\cos\theta/2 - \sin\theta/2)}{(\cos\theta/2 + \sin\theta/2)^2}$

$= \frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2 + \sin\theta/2}$

$= \frac{\cancel{\cos\theta/2}(1 - \tan\theta/2)}{\cancel{\cos\theta/2}(1 + \tan\theta/2)}$

$\cot\phi_1 = \cot\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$

$\boxed{\phi_1 = \frac{\pi}{4} + \frac{\theta}{2}}$

$r = a(1-\cos\theta)$

take log on b.s

$\log r = \log[a(1-\cos\theta)]$

$\log r = \log a + \log(1-\cos\theta)$

diff w.r.t θ

$\frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta} \times \sin\theta$

$\cot\phi_2 = \frac{\sin\theta}{1-\cos\theta}$

$= \frac{\cancel{2\sin\theta/2}\cos\theta/2}{2\sin^2\theta/2}$

$= \frac{\cos\theta/2}{\sin\theta/2}$

$\cot\phi_2 = \cot\theta/2$

$\boxed{\phi_2 = \frac{\theta}{2}}$

Angle b/w two curves

$$|\phi_1 - \phi_2| = \left|\frac{\pi}{4} + \frac{\theta}{2} - \frac{\theta}{2}\right|$$

$$= \frac{\pi}{4}.$$

$\therefore$ The curves intersect orthogonally.

9 $r = 6\cos\theta$ and $r = 2(1+\cos\theta)$

$r = 6\cos\theta$

take log on b.s

$\log r = \log(6\cos\theta)$

$\log r = \log 6 + \log\cos\theta$

diff w.r.t θ.

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos\theta} \times -\sin\theta$$

$$\cot\phi_1 = \frac{-\sin\theta}{\cos\theta}$$

$$\cot\phi_1 = -\tan\theta$$

$$\cot\phi_1 = \cot\left(\frac{\pi}{2} + \theta\right)$$

$$\phi_1 = \frac{\pi}{2} + \theta.$$

$r = 2(1+\cos\theta)$

take log on bs

$\log r = \log[2(1+\cos\theta)]$

$\log r = \log 2 + \log(1+\cos\theta)$

diff w.r.t θ.

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\cos\theta} \times -\sin\theta$$

$$\cot\phi_2 = \frac{-\sin\theta}{1+\cos\theta}$$

$$= \frac{-\cancel{2}\sin\theta/2 \cdot \cancel{\cos\theta/2}}{\cancel{2}\cos^2\theta/2}$$

$$= -\tan\theta/2$$

$$\cot\phi_2 = \cot\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\phi_2 = \frac{\pi}{2} + \frac{\theta}{2}$$

$$|\phi_1 - \phi_2| = \left|\left(\frac{\cancel{\pi}}{\cancel{2}} + \theta\right) - \left(\frac{\cancel{\pi}}{\cancel{2}} + \frac{\theta}{2}\right)\right|$$

$$= \frac{\theta}{2} \rightarrow (1)$$

To find θ

$r = 6\cos\theta$ & $r = 2(1+\cos\theta)$

$\Rightarrow \overset{3}{\cancel{6}}\cos\theta = \overset{1}{\cancel{2}}(1+\cos\theta)$

$3\cos\theta = 1+\cos\theta$

$2\cos\theta = 1$

$\cos\theta = 1/2$

$\theta = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta = \frac{\pi}{3}$

Substitute in eq^n 1

$|\phi_1 - \phi_2| = \frac{\pi}{6}$

10. $r = a\log\theta$ & $r = \frac{a}{\log\theta}$

$r = a\log\theta$

$\log r = \log[a\log\theta]$

$\log r = \log a + \log(\log\theta)$

diff w.r.t θ

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{\log\theta} \times \frac{1}{\theta} \times 1$

$\cot\phi_1 = \frac{1}{\theta\log\theta}$

$\tan\phi_1 = \theta\log\theta$

$r = \frac{a}{\log\theta}$

$\log r = \log\left(\frac{a}{\log\theta}\right)$

$= \log a - \log(\log\theta)$

diff w.r.t θ.

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 - \frac{1}{\log\theta} \times \frac{1}{\theta} \times 1$

$\cot\phi_2 = \frac{-1}{\theta\log\theta}$

$\tan\phi_2 = -\theta\log\theta$

Since ϕ_1 and ϕ_2 Cannot be obtain explicitly, we shall use the following formula.

$$|\tan(\phi_1 - \phi_2)| = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \cdot \tan\phi_2}$$

$$= \frac{\theta\log\theta - (-\theta\log\theta)}{1 + (\theta\log\theta \times -\theta\log\theta)}$$

$$= \frac{2\theta\log\theta}{1 - \theta^2(\log\theta)^2} \rightarrow (1)$$

To find θ

$r = a\log\theta$ & $r = \frac{a}{\log\theta}$

$\cancel{a}\log\theta = \frac{\cancel{a}}{\log\theta}$

$(\log\theta)^2 = 1$

$\log\theta = 1$

$\boxed{\theta = e^1}$

Substitute θ in eq^n ①

$\tan|\phi_1 - \phi_2| = \dfrac{2e\log_e e}{1 - e^2(\log_e e)^2}$ $\qquad \log_e e = 1$

$\tan|\phi_1 - \phi_2| = \dfrac{2e}{1-e^2}$

$|\phi_1 - \phi_2| = \tan^{-1}\left(\dfrac{2e}{1-e^2}\right)$

11. $r = \dfrac{a\theta}{1+\theta}$ & $r = \dfrac{a}{1+\theta^2}$ $\qquad \log\left(\dfrac{ab}{c}\right) = \log a + \log b - \log c$

$r = \dfrac{a\theta}{1+\theta}$	$r = \dfrac{a}{1+\theta^2}$
$\log r = \log\left(\dfrac{a\theta}{1+\theta}\right)$	$\log r = \log\left(\dfrac{a}{1+\theta^2}\right)$
$\log r = \log a + \log\theta - \log 1+\theta$	$\log r = \log a - \log(1+\theta^2)$
diff w.r.t θ	diff w.r.t θ.
$\dfrac{1}{r}\dfrac{dr}{d\theta} = 0 + \dfrac{1}{\theta} - \dfrac{1}{1+\theta} \times 1$	$\dfrac{1}{r}\dfrac{dr}{d\theta} = 0 - \dfrac{1}{1+\theta^2} \times 2\theta$
$\cot\phi_1 = \dfrac{1}{\theta} - \dfrac{1}{1+\theta}$	$\cot\phi_2 = \dfrac{-2\theta}{1+\theta^2}$
$\cot\phi_1 = \dfrac{1+\cancel{\theta}-\cancel{\theta}}{\theta(1+\theta)}$	$\boxed{\tan\phi_2 = \dfrac{-(1+\theta^2)}{2\theta}}$
$= \dfrac{1}{\theta(1+\theta)}$	
$\boxed{\tan\phi_1 = \theta(1+\theta)}$	

since ϕ_1 and ϕ_2 cannot be obtain explicitly, we shall use the following formula.

$|\tan(\phi_1 - \phi_2)| = \dfrac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \cdot \tan\phi_2}$

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$$= \frac{\theta(1+\theta)+\left(\frac{1+\theta^2}{2\theta}\right)}{1+\left[\theta(1+\theta)\times -\frac{(1+\theta^2)}{2\theta}\right]} \rightarrow ①$$

To find θ.

$r = \frac{a\theta}{1+\theta}$ & $r = \frac{a}{1+\theta^2}$

$$\frac{\cancel{a}\theta}{1+\theta} = \frac{\cancel{a}}{1+\theta^2}$$

$$\theta(1+\theta^2) = 1+\theta$$

$$\cancel{\theta}+\theta^3 = 1+\cancel{\theta}$$

$$\theta^3 = 1$$

$$\boxed{\theta = 1}$$

Sub in eq^n ①

$$= \frac{1(1+1)+\left(\frac{1+1^2}{2\times 1}\right)}{1+\left[1(1+1)\times\left(-\frac{1+1^2}{2\times 1}\right)\right]}$$

$$= \frac{1(2)+2/2}{1+\left[2\times\frac{-2}{2}\right]}$$

$$= \frac{2+1}{-1}$$

$|\tan(\phi_1-\phi_2)| = |-3| = 3$

$|\phi_1-\phi_2| = \tan^{-1}(3)$

12. $r(1+\cos\theta) = a$ and $r(1-\cos\theta) = b$.

$r(1+\cos\theta) = a$

take log on b.s.

$\log r + \log(1+\cos\theta) = \log a$.

diff. w.r.t θ.

$$\frac{1}{r}\cdot\frac{dr}{d\theta} + \frac{1}{1+\cos\theta}\times -\sin\theta = 0$$

$$\cot\phi_1 = \frac{\sin\theta}{1+\cos\theta}$$

$r(1-\cos\theta) = b$

take log on b.s

$\log r + \log(1-\cos\theta) = \log b$

diff. w.r.t. θ

$$\frac{1}{r}\cdot\frac{dr}{d\theta} + \frac{1}{1-\cos\theta}\times \sin\theta = 0$$

$$\cot\phi_2 = \frac{-\sin\theta}{1-\cos\theta}$$

$$\cot\phi_1 \cdot \cot\phi_2 = \frac{\sin\theta}{1+\cos\theta} \cdot \frac{-\sin\theta}{1-\cos\theta}$$

$$= \frac{-\sin^2\theta}{1^2-\cos^2\theta}$$

$$= -1.$$

13. $r^2\sin 2\theta = a^2$ and $r^2\cos 2\theta = b^2$

$r^2\sin 2\theta = a^2$

take log on b.s

$\log r^2 + \log\sin 2\theta = \log a^2$

diff. w.r.t θ

$$2\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\sin 2\theta} \cos 2\theta \times 2 = 0$$

$$\cancel{2}\cot\phi_1 = -\frac{\cancel{2}\cos 2\theta}{\sin 2\theta}$$

$$\cot\phi_1 = -\cot 2\theta$$

$r^2\cos 2\theta = b^2$

take log on b.s

$\log r^2 + \log\cos 2\theta = \log b^2$

diff. w.r.t θ

$$2\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\cos 2\theta} \times -\sin 2\theta \times 2 = 0$$

$$\cancel{2}\cot\phi_2 = \frac{\cancel{2}\sin 2\theta}{\cos 2\theta}$$

$$\cot\phi_2 = \tan 2\theta$$

$$\cot\phi_1 \cdot \cot\phi_2 = -\cot 2\theta \times \tan 2\theta$$

$$= -\cancel{\cot 2\theta} \times \frac{1}{\cancel{\cot 2\theta}}$$

$$= -1.$$

14. $r = 2\sin\theta$ and $r = 2\cos\theta$

$r = 2\sin\theta$

take log on b.s

$\log r = \log 2 + \log\sin\theta$

diff. w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sin\theta} \times \cos\theta$$

$$\cot\phi_1 = \cot\theta$$

$r = 2\cos\theta$

take log on b.s

$\log r = \log 2 + \log\cos\theta$

diff. w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos\theta} \times -\sin\theta$$

$$\cot\phi_2 = -\tan\theta$$

$$\cot\phi_1 \cdot \cot\phi_2 = \cot\theta \times -\tan\theta$$

$$= \cancel{\cot\theta} \times -\frac{1}{\cancel{\cot\theta}}$$

$$= -1$$

15. $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

$r = a(1-\cos\theta)$

take log on b.s

$\log r = \log a + \log(1-\cos\theta)$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta} \times \sin\theta$

$\cot\phi_1 = \frac{\sin\theta}{1-\cos\theta}$

$\cot\phi_1 = \cot\theta/2$

$\phi_1 = \theta/2$

$r = 2a\cos\theta$

take log on b.s

$\log r = \log 2a + \log\cos\theta$

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{\cos\theta} \times -\sin\theta$

$\cot\phi_2 = \frac{-\sin\theta}{\cos\theta}$

$\cot\phi_2 = -\tan\theta$

$\cot\phi_2 = \cot\left(\frac{\pi}{2}+\theta\right)$

$\phi_2 = \frac{\pi}{2} + \theta$

$|\phi_1 - \phi_2| = \left|\frac{\theta}{2} - \frac{\pi}{2}\right|$

$= \left|\frac{\theta - 2\theta}{2} - \frac{\pi}{2}\right|$

$= \left|\frac{\theta}{2} - \frac{\pi}{2}\right| \Rightarrow \frac{\pi}{2} + \frac{\theta}{2}$

$\cancel{a}(1-\cos\theta) = 2\cancel{a}\cos\theta$

$1-\cos\theta = 2\cos\theta$

$1 = 3\cos\theta$

$\theta = \cos^{-1}(1/3)$

16. $r^n = a^n\sec(n\theta+\alpha)$ and $r^n = b^n\sec(n\theta+\beta)$

$r^n = a^n\sec(n\theta+\alpha)$

take log on b.s.

$\log r^n = \log a^n + \log(\sec(n\theta+\alpha))$

$n\log r = \log a^n + \log(\sec(n\theta+\alpha))$

diff. w.r.t θ

$\cancel{n}\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{\sec(n\theta+\alpha)} \times \sec(n\theta+\alpha)\tan(n\theta+\alpha) \times \cancel{n}$

$\cot\phi_1 = 0 + \frac{\cancel{\sec(n\theta+\alpha)}\tan(n\theta+\alpha)}{\cancel{\sec(n\theta+\alpha)}}$

$\cot\phi_1 = \tan(n\theta+\alpha)$

$\cot\phi_1 = \cot\left(\frac{\pi}{2} - n\theta - \alpha\right)$

$\phi_1 = \left(\frac{\pi}{2} - n\theta + \alpha\right)$

$r^n = b^n \sec(n\theta+\beta)$

take log on b.s

$\log r^n = \log b^n + \log(\sec(n\theta+\beta))$

$n \log r = 0 + \log(\sec(n\theta+\beta))$

diff w.r.t θ

$\cancel{n} \cdot \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\sec(n\theta+\beta)} \times \sec(n\theta+\beta)\tan(n\theta+\beta) \times \cancel{n}$

$\cot\phi_2 = \frac{0+\sec(n\theta+\beta)\tan(n\theta+\beta)}{\sec(n\theta+\beta)}$

$\cot\phi_2 = \tan(n\theta+\beta)$

$\cot\phi_2 = \cot\left(\frac{\pi}{2} - n\theta - \beta\right)$

$\phi_2 = \frac{\pi}{2} - n\theta + \beta$

$\phi = |\phi_1 - \phi_2|$

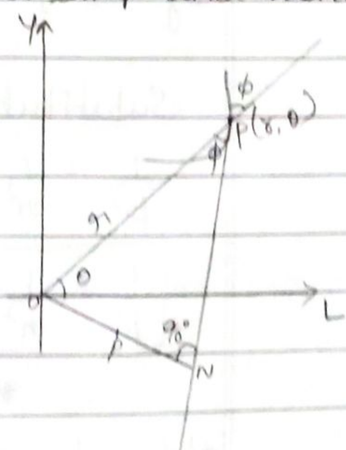
$= \left|\cancel{\frac{\pi}{2}} - \cancel{n\theta} - \alpha - \cancel{\frac{\pi}{2}} + \cancel{n\theta} + \beta\right|$

$\phi = |\beta - \alpha|$ (or) $\alpha - \beta$

9/12/22

** imp Prove that: Length of the Perpendicular from the pole of the tangent (or)

Prove with usual notations: $p = r\sin\phi$ and hence prove that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$



Proof: Let 'O' be the origin

OL be the initial line.

Let $P(r,\theta)$ be any point on the curve $r = f(\theta)$

$\therefore OP = r$, the radius vector

$L\hat{O}P = \theta$, the polar angle

Draw $ON = p$ (say), a perpendicular from the pole on the tangent at P

Let ϕ be the angle made by the radius vector and the tangent.

From fig.
$O\hat{N}P = 90°$,
From the right angled ΔONP we have,
$\sin\phi = \frac{opp}{hyp} = \frac{p}{r}$

$\Rightarrow \boxed{p = r\sin\phi}$

This is the expression for length of the Perpendicular.
To express p in terms of θ

Consider $p = r\sin\phi$
Squaring on both sides.
$$p^2 = r^2\sin^2\phi$$
take reciprocal on both sides.
$$\frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$$
$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{Cosec}^2\phi$$
$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi) \rightarrow (1)$$
But WKT, angle b/w radius vector & tangent is given by $\cot\phi = \frac{1}{r}\cdot\frac{dr}{d\theta}$

Substituting in eq^n (1)
$$\frac{1}{p^2} = \frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$
$$\boxed{\frac{1}{p^2} = \frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$$

Note: $p = r\sin\phi$ is called the pedal equation (or) $p-r$ equation of the curve $r = f(\theta)$

Find the Pedal equation for the following Curves :-

① $r = ae^{\theta \cot\alpha}$

Soln: $r = ae^{\theta\cot\alpha}$

$\log r = \log [ae^{\theta\cot\alpha}]$

$= \log a + \log e^{\theta\cot\alpha}$

$= \log a + \theta \cot\alpha \cdot \log e$

$\log r = \log a + \theta \cot\alpha$

diff. w.r.t θ.

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \cot\alpha \cdot 1$

$\cot\phi = \cot\alpha$

$\phi = \alpha$.

Pedal eqn : $p = r\sin\phi$

$\boxed{p = r\sin\alpha}$ is the pedal eqn (or) p-r eqn.

② $r = \frac{2a}{1+\cos\theta}$ → ①

$\log r = \log\left(\frac{2a}{1+\cos\theta}\right)$

$\log r = \log 2a - \log(1+\cos\theta)$

diff. w.r.t θ.

$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 - \frac{1}{1+\cos\theta} \times -\sin\theta$

$\cot\phi = \frac{\sin\theta}{1+\cos\theta}$

Pedal eqn is :- $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

$= \frac{1}{r^2}\left[1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}\right]$

$= \frac{1}{r^2}\left[\frac{(1+\cos\theta)^2 + \sin^2\theta}{(1+\cos\theta)^2}\right]$

$$= \frac{1}{r^2}\left[\frac{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}{(1+\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{1+1+2\cos\theta}{(1+\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{2+2\cos\theta}{(1+\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2}\left[\frac{\cancel{1+\cos\theta}}{(1+\cos\theta)^{\cancel{2}}}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2} \times \frac{1}{(1+\cos\theta)} \rightarrow (2)$$

To eliminate θ.

using (1), $\frac{1}{1+\cos\theta} = \frac{r}{2a}$.

Sub in (2)

$$\frac{1}{p^2} = \frac{\cancel{2}}{r^{\cancel{2}}} \times \frac{\cancel{r}}{\cancel{2}a}$$

$$\frac{1}{p^2} = \frac{1}{ra}.$$

$$p^2 = ar$$

$\boxed{p = \sqrt{ar}}$ is the pedal eq^n or p-r eq^n

(3) $r^m \cos m\theta = a^m \rightarrow (1)$

$$\log(r^m \cos m\theta) = \log(a^m)$$

$$\log r^m + \log(\cos m\theta) = \log(a^m)$$

$$m.\log r + \log(\cos m\theta) = \log a^m$$

diff. w.r.t θ.

$$m\frac{1}{r}\cdot\frac{dr}{d\theta} + \frac{1}{\cos m\theta} \times -\sin m\theta \times m$$

$$\cancel{m}.\cot\phi = \cancel{m}\tan m\theta$$

$$\cot\phi = \tan m\theta$$

Pedal eq^n :- $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

$$= \frac{1}{r^2}\left[1+\tan^2 m\theta\right] \qquad (\because 1+\tan^2\theta = \sec^2\theta)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\sec^2 m\theta \rightarrow (2)$$

To eliminate θ.

using (1) $r^m \cos m\theta = a^m$.

$$\frac{r^m}{a^m} = \frac{1}{\cos m\theta} = \sec m\theta.$$

Sub in (2)

$$\frac{1}{p^2} = \frac{1}{r^2} \times \left(\frac{r^m}{a^m}\right)^2.$$

$$\frac{1}{p^2} = \frac{r^{2m}}{r^2 \times a^{2m}}$$

$$\frac{1}{p^2} = \frac{r^{2m-2}}{a^{2m}}$$

$\boxed{p^2 \cdot r^{2m-2} = a^{2m}}$ is the pedal eq^n or p-r eq^n.

(4) $r^m = a^m(\cos m\theta + \sin m\theta)$

$$\log(r^m) = \log\left[a^m(\cos m\theta + \sin m\theta)\right]$$

$$m\log(r) = \log a^m + \log(\cos m\theta + \sin m\theta)$$

diff. w.r.t θ.

$$m \cdot \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{(\cos m\theta + \sin m\theta)} \times -\sin m\theta \times m + \cos m\theta \times m$$

$$\cancel{m}\cot\phi = \frac{\cancel{m}(\cos m\theta - \sin m\theta)}{(\cos m\theta + \sin m\theta)}$$

$$\cot\phi = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}.$$

Pedal eq^n: $\dfrac{1}{p^2} = \dfrac{1}{r^2}(1+\cot^2\phi)$

$$= \frac{1}{r^2}\left[1 + \frac{(\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{(\cos m\theta + \sin m\theta)^2 + (\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{\cos^2 m\theta + \sin^2 m\theta + 2\cos m\theta \sin m\theta + \cos^2 m\theta + \sin^2 m\theta - 2\cos m\theta \sin m\theta}{(\cos m\theta + \sin m\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{2}{(\cos m\theta + \sin m\theta)^2}\right] \rightarrow (2)$$

To eliminate θ

using (1) $\cos m\theta + \sin m\theta = \frac{r^m}{a^m}$

Sub in (2)

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{2}{(r^m/a^m)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{2 \times a^{2m}}{r^{2m}}\right]$$

$$\frac{1}{p^2} = \frac{2 \times a^{2m}}{r^{2m+2}}$$

$2p^2 . a^{2m} = r^{2m+2}$. is the pedal eq^n or p-r eq^n.

(5) $r^2 = a^2 \cos 2\theta$.

$\log(r^2) = \log[a^2 \cos 2\theta]$

$2 . \log r = \log a^2 + \log \cos 2\theta$

diff. w.r.t θ

$$2 \frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\cos 2\theta} \times -\sin 2\theta \times 2.$$

$$\cancel{2}\cot\phi = -\tan 2\theta \times \cancel{2}$$

$$\cot\phi = -\tan 2\theta.$$

Pedal eq^n: $\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$

$$= \frac{1}{r^2}\left(1 + (1 - \tan^2 2\theta)\right)$$

$$= \frac{1}{r^2}(1 - \tan^2 2\theta) \rightarrow (3)$$

To eliminate θ

$$\cos 2\theta = \frac{r^2}{a^2}$$

$$= \frac{1}{r^2}\left[1-\left(\frac{r^2}{a^2}\right)^2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{a^4}{r^4} \times \frac{1}{r^2}$$

$$p^2a^4 = r^6$$

$$\sqrt{p^2a^4} = \sqrt{r^6} \Rightarrow pa^2 = r^3$$

6) $r^n = a(1+\cos n\theta) \quad \rightarrow (1)$

$$\log r^n = \log[a(1+\cos n\theta)]$$

$$n\log r = \log a + \log(1+\cos n\theta)$$

diff. w.r.t θ.

$$n\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1+\cos n\theta} \times -\sin n\theta \times n.$$

$$\cancel{n}\cot\phi = \frac{-\cancel{n}\sin n\theta}{1+\cos n\theta}$$

$$\cot\phi = \frac{-\sin n\theta}{1+\cos n\theta}.$$

Pedal eq^n : $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

$$= \frac{1}{r^2}\left[1+\frac{\sin^2 n\theta}{(1+\cos n\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{(1+\cos n\theta)^2+\sin^2 n\theta}{(1+\cos n\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1+\cos^2 n\theta+2\cos n\theta+\sin^2 n\theta}{(1+\cos n\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{2+2\cos n\theta}{(1+\cos n\theta)^2}\right]$$

$$= \frac{2}{r^2}\left[\frac{\cancel{(1+\cos n\theta)}}{(1+\cos n\theta)^{\cancel{2}}}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2} \times \frac{1}{1+\cos n\theta} \quad \rightarrow (2)$$

To eliminate θ.

from eq^n (1) : $r^n = a(1+\cos n\theta)$

$$1+\cos n\theta = \frac{r^n}{a}$$

Sub in eq^n (2)

$$\frac{1}{p^2} = \frac{2}{r^2} \times \frac{1}{r^n/a}.$$

$$\frac{1}{p^2} = \frac{2a}{r^{n+2}}.$$

$\boxed{2a \cdot p^2 = r^{n+2}}$. This is the pedal eq^n or p-r eq^n.

(7) $r^n \sec n\theta = a^n \rightarrow$ (1) (or) $r^n = a^n \cos n\theta$.

$\log(r^n \sec n\theta) = \log a^n$

$\log r^n + \log \sec n\theta = n \log a$.

$n . \log r + \log \sec n\theta = n . \log a$.

diff. w.r.t θ

$$n . \frac{1}{r} . \frac{dr}{d\theta} + \frac{1}{\cancel{\sec n\theta}} \times \cancel{\sec n\theta} \times \tan n\theta \times n = 0.$$

$n[\cot\phi + \tan n\theta] = 0.$

$\cot\phi = -\tan n\theta.$

$\cot^2\phi = \tan^2 n\theta$

pedal eq^n : $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

$$= \frac{1}{r^2}(1+\tan^2 n\theta)$$

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\tan^2 n\theta) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}[\sec^2\theta] \rightarrow (2)$$

To eliminate θ :

from (1) $\frac{1}{\sec n\theta} = \frac{r^n}{a^n}.$

$$\sec n\theta = \frac{a^n}{r^n}$$

Sub in eq^n (2)

$$\frac{1}{p^2} = \frac{1}{r^2} \times \left(\frac{a^n}{r^n}\right)^2$$

$$\frac{1}{p^2} = \frac{a^{2n}}{r^{2n+2}}.$$

$\boxed{a^{2n} \cdot p^2 = r^{2n+2}}$ This is the pedal eq^n or p-r eq^n.

$$\sqrt{a^{2n} \cdot p^2} = \sqrt{r^{2n+2}}.$$

$$r^{n+1} = a^n p.$$

(8) $r^n = \operatorname{sech} n\theta \rightarrow$ (1)

$$\log r^n = \log(\operatorname{sech} n\theta)$$

$$\cancel{n} \cdot \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\cancel{\operatorname{sech} n\theta}} \times -\cancel{\operatorname{sech} n\theta} \times \tanh n\theta \times \cancel{n}$$

$\frac{d}{d\theta}(\sec\theta) = \sec\theta \cdot \tan\theta.$

$$\cot\phi = -\tanh n\theta.$$

$$\cot^2\phi = \tanh^2 n\theta.$$

Pedal eq^n: $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

$$= \frac{1}{r^2}(1+\tanh^2 n\theta)$$

$(\because) 1-\tanh^2 x = \operatorname{sech}^2 x$

$1-\operatorname{sech}^2 x = \tanh^2 x$

$$= \frac{1}{r^2}\left[1+1-\operatorname{sech}^2 n\theta\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[2-\operatorname{sech}^2 n\theta\right] \rightarrow (2)$$

To eliminate θ:

from (1) $\operatorname{sech} n\theta = r^n$

$\operatorname{sech}^2 n\theta = r^{2n}.$

Substitute in eq^n (2)

$$\frac{1}{p^2} = \frac{1}{r^2}\left[2-r^{2n}\right]$$

$\boxed{r^2 = p^2(2-r^{2n})}$ This is the pedal eq^n or p-r eq^n.

9 $r = a + b\cos\theta$

$\log r = \log(a + b\cos\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{a+b\cos\theta} \times -\sin\theta\, b$

$\cot\phi = \frac{-b\sin\theta}{a+b\cos\theta}$

$\cot\phi^2 = \frac{b^2\sin^2\theta}{(a+b\cos\theta)^2}$

Pedal eq^n: $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{b^2\sin^2\theta}{(a+b\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{(a+b\cos\theta)^2 + b^2\sin^2\theta}{(a+b\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{a^2+b^2\cos^2\theta + 2ab\cos\theta + b^2\sin^2\theta}{(a+b\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{a^2+b^2+2ab\cos\theta}{(a+b\cos\theta)^2}\right]$$

from ①, $r = a + b\cos\theta$

$r - a = b\cos\theta$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{a^2+b^2+2a(r-a)}{(a+r-a)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{a^2+b^2+2ar-2a^2}{r^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{b^2-a^2+2ar}{r^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^4}(b^2-a^2+2ar)$$

10 $r(1-\cos\theta) = 2a$.

$\log[r(1-\cos\theta)] = \log 2a$

$\log r + \log(1-\cos\theta) = \log 2a$

diff. w.r.t θ.

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{(1-\cos\theta)}(\sin\theta) = 0$$

$$\cot\phi + \frac{\sin\theta}{(1-\cos\theta)} = 0$$

$$\cot\phi = \frac{-\sin\theta}{(1-\cos\theta)}$$

Pedal eq^n : $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

$$= \frac{1}{r^2}\left[1+\left(\frac{-\sin\theta}{(1-\cos\theta)}\right)^2\right]$$

$$= \frac{1}{r^2}\left[1+\frac{\sin^2\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{(1-\cos\theta)^2+\sin^2\theta}{(1-\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{1+\cos^2\theta-2\cos\theta+\sin^2\theta}{(1-\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{2-2\cos\theta}{(1-\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}2\left[\frac{1-\cos\theta}{(1-\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2}\left(\frac{1}{1-\cos\theta}\right)$$

To eliminate θ.

using ① :- $r(1-\cos\theta) = 2a$

$$(1-\cos\theta) = \frac{2a}{r}$$

$$\frac{1}{(1-\cos\theta)} = \frac{r}{2a}$$

$$\frac{1}{p^2} = \frac{\cancel{2}}{r^{\cancel{2}}}\left(\frac{\cancel{r}}{\cancel{2}a}\right)$$

$$\frac{1}{p^2} = \frac{1}{ar}$$

$$p^2 = ar$$

$p = \sqrt{ar}$ is the pedal eq^n of p-r eq^n

11. $\frac{l}{r} = 1 + e\cos\theta \rightarrow (1)$

$$\log\left[\frac{l}{r}\right] = \log(1+e\cos\theta)$$

$$\log l - \log r = \log(1+e\cos\theta)$$

diff w r t θ

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1+e\cos\theta} \times -e\sin\theta$$

$$+\cot\phi = \frac{+e\sin\theta}{1+e\cos\theta}$$

$$\cot\phi = \frac{e\sin\theta}{1+e\cos\theta}$$

pedal eq^n : $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

$$= \frac{1}{r^2}\left[\frac{1+e^2\sin^2\theta}{(1+e\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{(1+e\cos\theta)^2+e^2\sin^2\theta}{(1+e\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{1^2+e^2\cos^2\theta+2e\cos\theta+e^2\sin^2\theta}{(1+e\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{1+2e\cos\theta+e^2(\cos^2\theta+\sin^2\theta)}{(1+e\cos\theta)^2}\right]$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1+e^2+2e\cos\theta}{(1+e\cos\theta)^2}\right] \rightarrow (2)$$

To eliminate θ :

from eq^n (1) $1+e\cos\theta = \frac{l}{r}$

$$e\cos\theta = \frac{l}{r} - 1$$

Sub in eq^n (2)

$$\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{1+e^2+2\left(\frac{l}{r}-1\right)}{\left(\frac{l}{r}\right)^2}\right]$$

$$= \frac{1\, r^2}{r^2 l^2}\left[1+e^2+\frac{2l}{r}-2\right]$$

$$\frac{1}{p^2} = \frac{1}{l^2}\left[e^2 - 1 + \frac{2l}{r}\right]$$

$$\boxed{\frac{1}{p^2} = \frac{(e^2-1)}{l^2} + \frac{2}{lr}}$$ is the pedal eq^n or p-r eq^n

12. Show that the pedal eq^n of $r^n = a^n \sin n\theta + b^n \cos n\theta$ is in the form $p^2(a^{2n} + b^{2n}) = r^{2n+2}$.

$$r^n = a^n \sin n\theta + b^n \cos n\theta \rightarrow (1)$$

$$\log(r^n) = \log(a^n \sin n\theta + b^n \cos n\theta)$$

$$n \log r = \log(a^n \sin n\theta + b^n \cos n\theta)$$

diff w.r.t θ.

$$n \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1}{a^n \sin n\theta + b^n \cos n\theta} \times a^n \cos n\theta \times n - b^n \sin n\theta \times n$$

$$\cancel{n} \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{\cancel{n}(a^n \cos n\theta - b^n \sin n\theta)}{a^n \sin n\theta + b^n \cos n\theta}$$

$$\cot\phi = \frac{a^n \cos n\theta - b^n \sin n\theta}{a^n \sin n\theta + b^n \cos n\theta}$$

Pedal eq^n : $\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{(a^n \cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{(a^n \sin n\theta + b^n \cos n\theta)^2 + (a^n \cos n\theta - b^n \sin n\theta)^2}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{a^{2n}\sin^2 n\theta + b^{2n}\cos^2 n\theta + \cancel{2a^n \sin n\theta\, b^n \cos n\theta} + a^{2n}\cos^2 n\theta + b^{2n}\sin^2 n\theta \cancel{- 2a^n b^n \cos n\theta \sin n\theta}}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{a^{2n} + b^{2n}}{(a^n \sin n\theta + b^n \cos n\theta)^2}\right] \rightarrow (2)$$

To eliminate θ :

from eq^n (1) : $\frac{1}{p^2} = \frac{1}{r^2}\left[\frac{a^{2n} + b^{2n}}{(r^n)^2}\right]$

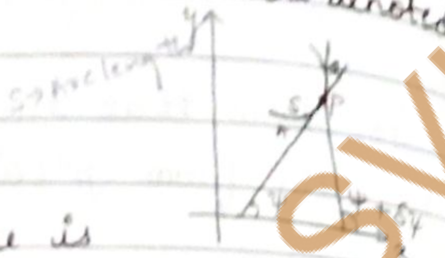
$$\frac{1}{p^2} = \frac{a^{2n} + b^{2n}}{r^{2n+2}}$$

$p^2(a^{2n} + b^{2n}) = r^{2n+2}$ is the pedal eq^n of p-r eq^n

Curvature and Radius of Curvature

Def: The Rate of bending of a curve C at a particular point P is called the curvature denoted by $\mathcal{K}$ (Kapa).

$$\mathcal{K} = \frac{d\psi}{ds} = \lim_{\delta s \to 0} \frac{\delta\psi}{\delta s}$$



The reciprocal of the curvature is called the radius of curvature denoted by ρ (rho)

$$\therefore \rho = \frac{1}{\mathcal{K}} = \frac{ds}{d\psi}$$

Note: Formulae connected with derivatives of Arc length

① Cartesian curve $y = f(x)$

(i) $\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$

(ii) $\sin\psi = \frac{dy}{ds}$, $\cos\psi = \frac{dx}{ds}$

$\therefore \tan\psi = \frac{dy}{dx}$ = Slope of the tangent

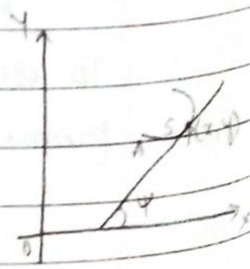
② Polar curve $r = f(\theta)$

(i) $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$

(ii) $\sin\phi = r\frac{d\theta}{ds}$; $\cos\phi = \frac{dr}{ds}$

Radius of curvature in the Cartesian form. (or) Obtain the Radius of curvature in the Cartesian form.

Proof:- Let $y = f(x)$ be the equation of the Cartesian curve and A be a fixed point on it.



Let $P(x, y)$ be a point on the curve such that $\widehat{AP} = s$

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Let ψ be the angle made by the tangent at 'P' with the x-axis.

WKT. $\tan\psi = \frac{dy}{dx}$ = slope of the tangent.

diff. w.r.t. s.

$$\frac{d}{ds}(\tan\psi) = \frac{d}{ds}\left(\frac{dy}{dx}\right)$$ $\tan\psi = \sec^2\psi$

$\times$ly & $\div$ by dx

$$\sec^2\psi \cdot \frac{d\psi}{ds} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds}$$

$$\sec^2\psi \cdot \frac{1}{\rho} = \frac{d^2y}{dx^2} \cdot \cos\psi$$

$$\frac{\sec^2\psi}{\cos\psi} \cdot \frac{1}{\rho} = \frac{d^2y}{dx^2}$$

$$\sec^3\psi \cdot \frac{1}{\rho} = \frac{d^2y}{dx^2}$$

$$\frac{\sec^3\psi}{d^2y/dx^2} = \rho$$

$$\rho = \frac{(\sec^2\psi)^{3/2}}{d^2y/dx^2}$$ $\sec^2\psi = 1+\tan^2\psi$

$$\rho = \frac{(1+\tan^2\psi)^{3/2}}{d^2y/dx^2}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Let $\frac{dy}{dx} = y_1$; $\frac{d^2y}{dx^2} = y_2$

$$\boxed{\rho = \frac{[1+y_1^2]^{3/2}}{y_2}}$$ This is the expression for R.O.C in the cartesian form.

Note :- Sometimes y_1 at some point on the curve becomes infinity (i.e when the tangent is $\perp^r$ to the x-axis, $\tan\psi = \tan 90^\circ = \infty$) in this case we cannot apply the formula for ρ in the above form. Alternate formula for ρ is $\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$ when $x_1 = \frac{dx}{dy}$; $x_2 = \frac{d^2x}{dy^2}$

① ~~Find~~ Show that the radius of curvature of the curve $y = c.\cosh\left(\frac{x}{c}\right)$ is $\frac{y^2}{c}$, where c is a constant

$$y = c.\cosh\left(\frac{x}{c}\right) \rightarrow ①$$

diff. w.r.t x

$$y_1 = \cancel{c}.\sinh\left(\frac{x}{c}\right)\cdot\frac{1}{\cancel{c}} \qquad \left(\because \frac{d}{d\theta}(\cosh\theta) = \sinh\theta\right)$$

$$\boxed{y_1 = \sinh\left(\frac{x}{c}\right)}$$

diff. w.r.t x

$$\boxed{y_2 = \cosh\left(\frac{x}{c}\right)\cdot\frac{1}{c}} \qquad \left(\because \frac{d}{d\theta}(\sinh\theta) = \cosh\theta\right)$$

R.o.c ; $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$\rho = \frac{\left[1+\sinh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\frac{1}{c}\cdot\cosh\left(\frac{x}{c}\right)}$$

$$\rho = \frac{c.\left[\cosh^2\left(\frac{x}{c}\right)\right]^{3/2}}{\cosh\left(\frac{x}{c}\right)} \qquad \left(\because \cosh^2\theta - \sinh^2\theta = 1 \Rightarrow \cosh^2\theta = 1+\sinh^2\theta\right)$$

$$= \frac{c.\cosh^3\left(\frac{x}{c}\right)}{\cosh\left(\frac{x}{c}\right)}$$

$$\rho = c.\cosh^2\left(\frac{x}{c}\right) \rightarrow ②$$

from ①, $\cosh\left(\frac{x}{c}\right) = \frac{y}{c}$

Sub this in ②

$$\rho = c.\frac{y^2}{c^2}$$

$$\boxed{\rho = \frac{y^2}{c}}$$

② Find the radius of curvature of the curve $y = a\log\left(\sec\left(\frac{x}{a}\right)\right)$ is $a\sec\left(\frac{x}{a}\right)$

$$y = a\log\left(\sec\left(\frac{x}{a}\right)\right)$$

diff. w.r.t x.

$$y_1 = \cancel{a} \cdot \frac{1}{\sec\left(\frac{x}{a}\right)} \times \sec\left(\frac{x}{a}\right) \cdot \tan\left(\frac{x}{a}\right) \cdot \frac{1}{\cancel{a}}.$$

$$\boxed{y_1 = \tan\left(\frac{x}{a}\right)}$$

diff. w.r.t x.

$$\boxed{y_2 = \sec^2\left(\frac{x}{a}\right) \cdot \frac{1}{a}}.$$

R.O.C ; $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$\rho = \frac{\left[1+\tan^2\left(\frac{x}{a}\right)\right]^{3/2}}{\sec^2(x/a) \cdot \frac{1}{a}}.$$

$$\rho = \frac{a \cdot \left[\sec^2\left(\frac{x}{a}\right)\right]^{3/2}}{\sec^2(x/a)}$$

$$\rho = \frac{a \cdot \left[\sec^3\left(\frac{x}{a}\right)\right]}{\sec^2\left(\frac{x}{a}\right)}$$

$$\boxed{\rho = a \cdot \sec\left(\frac{x}{a}\right)}$$

③ Find the radius of curvature of $y = 4\sin x - \sin 2x$ at $x = \pi/2$

$$y = 4\sin x - \sin 2x.$$

diff. w.r.t x.

$$y_1 = 4\cos x - 2\cos 2x. \rightarrow ①$$

at $x = \pi/2$; $y_1 = 4\cos\left(\frac{\pi}{2}\right) - 2\cos\left(\frac{2\pi}{\cancel{2}}\right)$

$$= (4\times 0) - (2\times -1)$$

$$\boxed{y_1 = 2}$$

diff. w.r.t x.

$$y_2 = -4\sin x + 4\sin 2x.$$

at $x = \pi/2$; $y_2 = -4\sin(\pi/2) + 4\sin\left(\frac{2\pi}{\cancel{2}}\right)$

$$= (-4\times 1) + (4\times 0)$$

$$\boxed{y_2 = -4}$$

R.O.C ; $\rho = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+2^2)^{3/2}}{-4}$$

$$\rho = \frac{5^{3/2}}{-4} = \frac{\sqrt{125}}{-4}$$

$$|\rho| = \left|\frac{-5\sqrt{5}}{4}\right|$$

$$\boxed{|\rho| = \frac{5\sqrt{5}}{4}}$$

④ Find the radius of curvature for the curve $a^2y = x^3 - a^3$ at the point where the curve cuts x-axis

Since the curve cuts the x-axis, we have $y=0$

Substitute $y=0$ in the given curve.

i.e $a^2y = x^3 - a^3$

$y=0 \Rightarrow 0 = x^3 - a^3$

$x^3 = a^3$

$\boxed{x = a}$.

∴ To find R.O.C of the given curve at $(a,0)$

Consider $a^2y = x^3 - a^3$

diff. w.r.t x

$a^2 \cdot y_1 = 3x^2 - 0$

$y_1 = \frac{3x^2}{a^2} \rightarrow ①$

at $(a,0)$; $y_1 = \frac{3a^2}{a^2}$

$\boxed{y_1 = 3}$

diff eqn w.r.t x.

$y_2 = \frac{3}{a^2} \cdot 2x$

$y_2 = \frac{6x}{a^2}$

at $(a,0)$; $y_2 = \frac{6x}{a^2}$

$= \frac{6a}{a^2}$

$\boxed{y_2 = 6/a}$

$$\rho = \frac{(1+9)^{3/2}}{6/a}$$

$$\rho = \frac{a + 10^{3/2}}{6}$$

$$\rho = \frac{a \times \sqrt{1000}}{3}$$

$$\rho = \frac{a \times 10\sqrt{10}}{3}$$

$$\boxed{\rho = \frac{5a\sqrt{10}}{3}}$$

(5) Find the radius of curvature of the curve $y^2 = \frac{a^2(a-x)}{x}$ at the point $(a,0)$

$$y^2 = \frac{a^2(a-x)}{x}$$

$$xy^2 = a^3 - a^2x$$

diff. w.r.t x.

$$x \cdot 2yy_1 + y^2 \cdot 1 = 0 - a^2 \cdot 1$$

$$2xyy_1 = -a^2 - y^2$$

$$y_1 = \frac{-a^2 - y^2}{2xy}$$

at $(a,0) \Rightarrow \boxed{x_1 = 0}$.

$$(a^2+y^2)x_1 = -2xy$$

diff w.r.t y.

$$(a^2+y^2)\,x_2 + x_1\,(0+2y) = -2[x \cdot 1 + y \cdot x_1]$$

$$(a^2+y^2)\,x_2 = -2x - 2x_1y - 2x_1y$$

$$(a^2+y^2)x_2 = -2x - 4x_1y$$

$$x_2 = \frac{-2x - 4x_1y}{(a^2+y^2)}$$

$\therefore$ at $(a,0)$; $x_2 = \frac{-2a-0}{(a^2+0)} = -\frac{2a}{a^2}$

$$x_2 = -2/a$$

$$\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$$

$$= \frac{(1+0)^{3/2}}{-2/a}$$

$$= \frac{1 \times a}{-2}$$

$$|\rho| = \left|\frac{-a}{2}\right|$$

$$|\rho| = \frac{a}{2}$$

(13) $r = a(1+\cos\theta)$

$\log r = \log a + \log(1+\cos\theta)$

diff. w.r.t θ

$$\cot\phi = \frac{1}{1+\cos\theta} \times -\sin\theta$$

$$\cot\phi = \frac{-\sin\theta}{1+\cos\theta}$$

Pedal eq^n :- $\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$

$$= \frac{1}{r^2}\left[1 + \frac{\sin^2\theta}{(1+\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{(1+\cos\theta)^2 + \sin^2\theta}{(1+\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{1+\cos^2\theta + 2\cos\theta + \sin^2\theta}{(1+\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{2+2\cos\theta}{(1+\cos\theta)^2}\right]$$

$$= \frac{2}{r^2}\left[\frac{\cancel{1+\cos\theta}}{(1+\cos\theta)^{\cancel{2}}}\right]$$

$$\frac{1}{p^2} = \frac{2}{r^2} \times \frac{1}{1+\cos\theta} \rightarrow (2)$$

To eliminate θ

$r = a(1+\cos\theta)$

$1+\cos\theta = \frac{r}{a}$

sub in eq^n 2

$\frac{1}{p^2} = \frac{2}{r^2} \times \frac{1}{r/a}$

$\frac{1}{p^2} = \frac{2a}{r^3}$

$r^3 = p^2 \cdot 2a$.

14) $\frac{2a}{r} = 1-\sin\theta$.

$\log 2a - \log r = \log(1-\sin\theta)$

diff w.r.t θ.

$0 - \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1}{1-\sin\theta} \times -\cos\theta$

$+\cot\phi = \frac{+\cos\theta}{1-\sin\theta}$

$\cot\phi = \frac{\cos\theta}{1-\sin\theta}$.

Pedal eq^n :- $\frac{1}{p^2} = \frac{1}{r^2}\left[1+\cot^2\phi\right]$

$= \frac{1}{r^2}\left[1 + \frac{\cos^2\theta}{(1-\sin\theta)^2}\right]$

$= \frac{1}{r^2}\left[\frac{(1-\sin\theta)^2 + \cos^2\theta}{(1-\sin\theta)^2}\right]$

$= \frac{1}{r^2}\left[\frac{1^2+\sin^2\theta - 2\sin\theta + \cos^2\theta}{(1-\sin\theta)^2}\right]$

$= \frac{1}{r^2}\left[\frac{1+1-2\sin\theta}{(1-\sin\theta)^2}\right]$

$= \frac{1}{r^2}\left[\frac{2-2\sin\theta}{(1-\sin\theta)^2}\right]$

$$= \frac{1}{r^2}\left[\frac{2(1-\sin\theta)}{(1-\sin\theta)^2}\right]$$

$$= \frac{2}{r^2}\left[\frac{1}{1-\sin\theta}\right] \rightarrow (2)$$

To eliminate θ

from (1) $\rightarrow \frac{1}{1-\sin\theta} = \frac{r}{2a}$

Sub in eq^n (2)

$$\frac{1}{p^2} = \frac{2}{r^2}\left[\frac{r}{2a}\right]$$

$$= \frac{1}{ra}$$

$$p^2 = ar$$

$p = \sqrt{ar}$ is the pedal eq^n or p-r eq^n.

(6) Find the R.O.C of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $(a/2, a/2)$

$$\sqrt{x}+\sqrt{y}=\sqrt{a}.$$

diff w.r.t x.

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\cdot y_1 = 0$$

$$\frac{y_1}{2\sqrt{y}} = \frac{-1}{2\sqrt{x}}$$

$$y_1 = \frac{-\cancel{2}\sqrt{y}}{\cancel{2}\sqrt{x}} \rightarrow (1)$$

at $(a/2, a/2)$

$$y_1 = \frac{\sqrt{a/2}}{\sqrt{a/2}}$$

$$y_1 = -1$$

diff w.r.t x

$$y_2 = \frac{\sqrt{x}\cdot\frac{1}{2\sqrt{y}}\,y_1 - \sqrt{y}\,\frac{1}{2\sqrt{x}}}{x}$$

$$y_2 = \frac{\frac{\sqrt{a}}{2} \cdot \frac{1}{2\left(\sqrt{\frac{a}{2}}\right)}(-1) - \left(\frac{\sqrt{a}}{2}\right)}{a/2}.$$

$$y_2 = \frac{-1/2 - 1/2}{a/2}$$

$$y_2 = \frac{2}{a}[-1]$$

$$= +\frac{2}{a}$$

$$y_2 = +\frac{2}{a}.$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{[1+(-1)^2]^{3/2}}{+2/a}.$$

$$= \frac{[1+1]^{3/2}}{+2/a}$$

$$= \frac{(2)^{3/2}}{+2/a}$$

$$= \frac{a \cdot \sqrt{8}}{+2}$$

$$= \frac{a \cancel{2} \sqrt{2}}{+\cancel{2}}.$$

$$= +a\sqrt{2}.$$

$$|\rho| = |+a\sqrt{2}|$$

$$\rho = a\sqrt{2}.$$

(7) Find the R.O.C of parabola $y^2 = 4ax$ at (a, a)

$y^2 = 4ax.$

diff. w.r.t x

$2y \cdot y_1 = 4a$

$y_1 = \frac{4a}{2y}$ $\quad y_1 = 2a$

at (a, a)

$$y_1 = \frac{4a}{2a}$$

$$y_1 = 2.$$

diff w.r.t x.

$$y_2 = \frac{2y(0) - 4a(2y_1)}{(2y)^2}$$

$$y_2 = \frac{0 - 8a(y_1)}{4y^2}$$

$$y_2 = \frac{-8ay_1}{4y^2}$$

at (a,a)

$$y_2 = \frac{-8a(a)}{4a^2}$$

$$= \frac{-16a}{4a^2}$$

$$y_2 = \frac{-4}{a}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\rho = \frac{(1+2^2)^{3/2}}{4/a}$$

$$\rho = \frac{a(5)^{3/2}}{4}$$

$$\rho = \frac{a5\sqrt{5}}{4}.$$

(8) Find the radius of curvature of the curve $x^2y = a(x^2+y^2)$ at the point $(-2a, 2a)$

$$x^2y = a(x^2+y^2)$$

diff. w.r.t x

$$x^2 \cdot y_1 + y \cdot 2x = a(2x + 2y \cdot y_1)$$

$$x^2 \cdot y_1 + 2xy = 2ax + 2ayy_1$$

$$x^2y_1 - 2ayy_1 = 2ax - 2xy$$

$$y_1(x^2 - 2ay) = 2ax - 2xy$$

$\frac{dy}{dx} = y_1 = \frac{2ax - 2xy}{x^2 - 2ay}$

at $(-2a, 2a)$

$y_1 = \frac{(2a \times -2a) - (2x - 2a \times 2a)}{(-2a)^2 - (2a \times 2a)}$

$y_1 = \infty$

$\therefore$ Considering $x_1 = \frac{dx}{dy} = \frac{x^2 - 2ay}{2ax - 2xy}$

at $(-2a, 2a)$, $\boxed{x_1 = 0}$

Consider $(2ax - 2xy)x_1 = x^2 - 2ay$

diff w.r.t y

$(2a.x - 2xy)x_2 + x_1 \cdot \{2ax_1 - 2(x.1 + y.x_1)\} = 2x x_1 - 2a.1$

at $(-2a, 2a)$

$[(2a \times -2a) - (2x - 2a \times 2a)]x_2 + 0 = 0 - 2a.$

$(-4a^2 + 8a^2)x_2 = -2a$

$(4a^2)x_2 = -2a$

$x_2 = \frac{-2a}{4a^2}$

$\boxed{x_2 = \frac{-1}{2a}}$

R.O.C $\rho = \frac{(1 + x_1^2)^{3/2}}{x_2}$

$\rho = \frac{(1+0)^{3/2}}{-1/2a}$

$\rho = -2a \times 1$

$\boxed{|\rho| = 2a}$

* (9) Find the R.O.C of the curve $x^3 + y^3 = 3axy$ at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ (or) $x^3 + y^3 = 3xy$ at $\left(\frac{3}{2}, \frac{3}{2}\right)$

$x^3 + y^3 = 3axy$

diff. w.r.t x.

$$3x^2 + 3y^2 \cdot y_1 = 3a[x \cdot y_1 + y \cdot 1]$$
$$\cancel{3}[x^2 + y^2 \cdot y_1] = \cancel{3}a[x \cdot y_1 + y \cdot 1]$$
$$y^2 \cdot y_1 - axy_1 = -x^2 + ay \cdot 1$$
$$(y^2 - ax)\, y_1 = ay - x^2$$
$$y_1 = \frac{ay - x^2}{y^2 - ax} \rightarrow (1)$$

at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$

$$y_1 = \frac{a \times 3a/2 - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a \times \frac{3a}{2}}$$
$$= \frac{\frac{3a^2}{2} - \frac{9a^2}{2}}{\frac{9a^2}{2} - \frac{3a^2}{2}}$$
$$= -\frac{\cancel{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)}}{\cancel{\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right)}}$$
$$\boxed{y_1 = -1.}$$

$$y_1 = \frac{ay - x^2}{y^2 - ax}$$
$$y_1(y^2 - ax) = ay - x^2$$
$$y_1 y^2 - axy_1 = ay - x^2$$
$$y_1(y^2 - ax) = ay - x^2$$

diff w.r.t x.

$$(y^2 - ax)y_2 + y_1(2y \cdot y_1 - a) = ay_1 - 2x$$
$$(y^2 - ax)\, y_2 + 2y \cdot y_1^2 - ay_1 = ay_1 - 2x$$
$$(y^2 - ax)\, y_2 = ay_1 - 2x - 2y \cdot y_1^2 - ay_1$$
$$(y^2 - ax)\, y_2 = 2a \cdot y_1 - 2x - 2yy_1^2$$
$$(y^2 - ax)\, y_2 = 2ay_1 - 2x - 2yy_1^2$$

at $(3a/2, 3a/2)$

$$\left[\left(\frac{3a}{2}\right)^2 - a \times \frac{3a}{2}\right] y_2 = (2a \times -1) - \cancel{2} \times \frac{3a}{\cancel{2}} - \cancel{2} \times \frac{3a}{\cancel{2}} \times (-1)^2$$
$$\left(\frac{9a^2}{4} - \frac{3a^2}{2}\right) y_2 = -2a - 3a - 3a.$$
$$\left(\frac{9a^2 - 6a^2}{4}\right) y_2 = -8a$$

$$\frac{3a^2}{4} \cdot y_2 = -8a.$$

$$y_2 = -8\cancel{a} \times \frac{4}{3a^{\cancel{2}}}$$

$$y_2 = \frac{-32}{3a}.$$

R.O.C, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+1)^{3/2}}{-32/3a}$$

$$= \frac{3a \times 2^{3/2}}{-32.}$$

$$= \frac{3a \times 2\cancel{\sqrt{2}}}{-16 \times \sqrt{2} \times \cancel{\sqrt{2}}}.$$

$$= \frac{3a \times \cancel{2}^{1}}{-_{8}\cancel{16} \times \sqrt{2}}$$

$$|\rho| = \left|\frac{-3a}{8\sqrt{2}}\right|$$

$$|\rho| = \frac{3a}{8\sqrt{2}}$$

Radius of Curvature in the Parametric form.

Consider the Parametric curve $x = x(t), y = y(t)$

R.O.C in the Parametric form is given by

$$\rho = \frac{[(x')^2 + (y')^2]^{3/2}}{x'y'' - y'x''}$$

Note: For convient we will be applying R.O.C in the cartesian form only. given by.

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} \quad \text{where } y_1 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

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(10) Find the R.O.C for the curve $x^4+y^4=2$ at $(1,1)$

$$x^4+y^4=2$$

$$4x^3+4y^3y_1=0$$

$$y_1=\frac{-\cancel{4}x^3}{\cancel{4}y^3}$$

$$y_1=\left(\frac{-1^3}{1^3}\right)$$

$$\boxed{y_1=-1}$$

$$y_1=\frac{-x^3}{y^3}$$

$$y_2=\frac{-x^3(3y^2y_1)-y^3(-3x^2)}{y^3}$$

$$=\frac{-1(3(-1))-1(-3(1))}{1}$$

$$y_2=\frac{-3-3}{1}$$

$$\boxed{y_2=-6}$$

$$\rho=\frac{(1+(y_1)^2)^{3/2}}{y_2}$$

$$=\frac{(1+(-1)^2)^{3/2}}{6}$$

$$=\frac{\sqrt{2^3}}{6}$$

$$=\frac{\cancel{2}\sqrt{2}}{\cancel{6}\,3}$$

$$\boxed{\rho=\frac{\sqrt{2}}{3}}$$

Find the R.O.C for the following Curves.

(1) $x = a\log(\sec t + \tan t)$; $y = a\sec t$

diff. w.r.t. t.

$\frac{dx}{dt} = \frac{a(\sec t.\tan t + \sec^2 t)}{(\sec t + \tan t)}$

$\frac{dy}{dt} = a\sec t.\tan t.$

$= \frac{a\sec t(\cancel{\tan t + \sec t})}{(\cancel{\sec t + \tan t})}$

$\frac{dx}{dt} = a\sec t$

Consider $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

$= \frac{\cancel{a\sec t}.\tan t}{\cancel{a\sec t}}$

$\boxed{y_1 = \tan t}$

diff. w.r.t. x.

$y_2 = \sec^2 t . \frac{dt}{dx}$

$= \sec^{\cancel{2}} t . \frac{1}{a\cancel{\sec t}}$

$\boxed{y_2 = \frac{1}{a}\sec t}$

R.O.C, $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$= \frac{(1+\tan^2 t)^{3/2}}{\frac{1}{a}\sec t}$

$= \frac{(\sec^2 t)^{3/2}}{\frac{1}{a}\sec t}$

$= \frac{a.\sec^3 t}{\sec t}.$

$\boxed{\rho = a\sec^2 t}$

(2) $x = a\cos^3\theta \quad ; \quad y = a\sin^3\theta \quad$ at $\theta = \pi/4$

diff w.r.t θ.

$\frac{dx}{d\theta} = a\,3\cos^2\theta \times -\sin\theta \qquad \frac{dy}{d\theta} = a\,3\sin^2\theta \times \cos\theta$

$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \qquad \frac{dy}{d\theta} = 3a\sin^2\theta \cdot \cos\theta$

Consider $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$= \frac{3a\sin^2\theta \cdot \cos\theta}{-3a\cos^2\theta \cdot \sin\theta}$

$y_1 = \boxed{\frac{dy}{dx} = -\tan t}$

diff w.r.t x

$y_2 = -\sec^2\theta \cdot \frac{d\theta}{dx}$

$= \frac{+\sec^2\theta}{+3a\cos^2\theta\sin\theta}$

$\boxed{y_2 = \frac{1}{3a} \times \sec^4\theta \times \operatorname{Cosec}\theta}$

at $\theta = \pi/4$: $y_1 = -\tan\frac{\pi}{4}$

$y_1 = -1$

$y_2 = \frac{1}{3a} \times \sec^4\left(\frac{\pi}{4}\right) \times \operatorname{Cosec}\left(\frac{\pi}{4}\right)$

$= \frac{1}{3a} \times (\sqrt{2})^4 \times \sqrt{2}$

$= \frac{1}{3a} \times ((\sqrt{2})^2)^2 \times \sqrt{2}$

$y_2 = \frac{4\sqrt{2}}{3a}$

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R.O.C $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$= \frac{(1+1)^{3/2}}{\frac{4\sqrt{2}}{3a}}$

$= \frac{3a \times 2^{3/2}}{4\sqrt{2}}$

$= \frac{2\cancel{\sqrt{2}} \times 3a}{{}_2\cancel{4}\cancel{\sqrt{2}}}$

$\boxed{\rho = \frac{3a}{2}}$

③ $x = a(\theta + \sin\theta)$; $y = a(1-\cos\theta)$

diff. w.r.t θ.

$\frac{dx}{d\theta} = a(1+\cos\theta)$

$\frac{dy}{d\theta} = a(0-(-\sin\theta))$

$\frac{dy}{d\theta} = a\sin\theta$

consider. $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a}\sin\theta}{\cancel{a}(1+\cos\theta)}$

$= \frac{\cancel{2}\sin\theta/2 \cdot \cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$

$\boxed{y_1 = \tan\theta/2}$

diff. w.r.t x.

$y_2 = \sec^2\theta/2 \cdot 1/2 \; \frac{d\theta}{dx}$

$= \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)\frac{1}{a(1+\cos\theta)}$

$= \frac{1}{2a}\frac{\sec^2(\theta/2)}{2\cos^2(\theta/2)}$

$\boxed{y_2 = \frac{1}{4a}\sec^4(\theta/2)}$

R.O.C ; $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$= \frac{(1+(\tan\theta/2)^2)^{3/2}}{\frac{1}{4a}\sec^4(\theta/2)}$

$$= \frac{[\sec^2(\theta/2)]^{3/2}}{\frac{1}{4a}\sec^4(\theta/2)}$$

$$= \frac{4a\sec^2(\theta/2)}{\sec^4(\theta/2)}$$

$$\rho = \frac{4a}{\sec^3(\theta/2)}$$

$$\boxed{\rho = 4a\cos(\theta/2)}$$

(4) $x = a[\cos t + \log\tan t/2]$; $y = a\sin t$

diff w.r.t t

$$\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan t/2} \times \sec^2 t/2 \times \frac{1}{2}\right]$$

$$= a\left[-\sin t + \frac{1}{2}\cdot\frac{1}{\cos^2(t/2)}\,\frac{\cos(t/2)}{\sin(t/2)}\right]$$

$$= a\left[-\sin t + \frac{1}{2}\,\frac{1}{\sin(t/2)\cos(t/2)}\right]$$

$$= a\left[-\sin t + \frac{1}{\sin t}\right]$$

$$\frac{dx}{dt} = a\left[\frac{-\sin^2 t + 1}{\sin t}\right]$$

$$= a\left[\frac{\cos^2 t}{\sin t}\right]$$

$$= a\cos^2 t \cdot \operatorname{cosec} t$$

$$y = a\sin t$$

$$\frac{dy}{dt} = a\cos t$$

Consider $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{\cancel{a}\cos t}{\cancel{a}\cos^2 t\,\operatorname{cosec} t}$

$$= \frac{1}{\cos t\,\operatorname{cosec} t}$$

$$\frac{dy}{dx} = \frac{\sin t}{\cos t}$$

$$y_1 = \tan t$$

diff w.r.t x.

$$y_2 = \sec^2 t \cdot \frac{dt}{dx}$$

$$= \frac{\sec^2 t}{a \operatorname{cosec} t \cos^2 t}$$

$$y_2 = \frac{1}{a} \sec^4 t \cdot \sin t$$

R.O.C. $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$= \frac{(1+\tan^2 t)^{3/2}}{\frac{1}{a}\sec^4 t \cdot \sin t}$$

$$= \frac{a \cdot (\sec^2 t)^{3/2}}{\sec^4 t \cdot \sin t}$$

$$= \frac{a \cdot \sec^3 t}{\sec^4 t \cdot \sin t}$$

$$= a \cdot \frac{1}{\sec t \sin t}$$

$$= a \frac{\cos t}{\sin t}$$

$$\rho = a \cot t$$

⑤ $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

diff w.r.t t

$\frac{dx}{dt} = a(-\sin t + t\cos t + \sin t)$	$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t)$
$\frac{dx}{dt} = at\cos t$	$\frac{dy}{dt} = at\sin t$

Consider $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cancel{at}\sin t}{\cancel{at}\cos t}$

$$\boxed{y_1 = \tan t}$$

diff w.r.t x

$$y_2 = \sec^2 t \frac{dt}{dx}$$

$$= \sec^2 t \times \frac{1}{at\cos t}$$

$$\boxed{y_2 = \frac{\sec^3 t}{at}}$$

$$\text{R.O.C : } \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

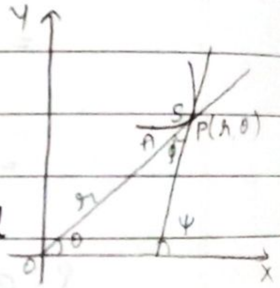
$$= \frac{(1+\tan^2 t)^{3/2}}{\frac{\sec^3 t}{at}}$$

$$= \frac{(1+\cancel{\sec^3 t})^{3/2}}{\frac{\cancel{\sec^3 t}}{at}}$$

$$\boxed{\rho = at}$$

** → An expression for R.O.C in the Polar form.

Derive the R.O.C in the Polar form.



Proof :- Let $r = f(\theta)$ be the equation of the polar curve

Let $OP = r$ be the radius vector and ϕ be the angle between the radius vector and the tangent at $P(r, \theta)$

Let ψ be the angle made by the tangent at P with the initial line.

Let A be a fixed point on the curve so that arc $\overset{\frown}{AP} = s$

WKT. $\psi = \theta + \phi$ (∵ Exterior angle = Sum of the interior angles)

diff w.r.t s.

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$\frac{1}{\rho} = \frac{d\theta}{ds}\left(1 + \frac{d\phi}{d\theta}\right)$$

$$\frac{ds/d\theta}{\left(1+\frac{d\phi}{d\theta}\right)} = \rho$$

i.e. $\rho = \dfrac{ds/d\theta}{\left(1+\frac{d\phi}{d\theta}\right)} \rightarrow (1)$

W.K.T. $\tan\phi = r\dfrac{d\theta}{dr}$

Let $\dfrac{dr}{d\theta} = r_1$

$\tan\phi = \dfrac{r}{r_1}$

diff w.r.t θ.

$\dfrac{vu' - uv'}{v^2}$

$\sec^2\phi \cdot \dfrac{d\phi}{d\theta} = \dfrac{r_1 r_1 - r r_2}{r_1^2}$ [u/v rule]

$\dfrac{d\phi}{d\theta} = \dfrac{r_1^2 - r r_2}{r_1^2 \sec^2\phi}$

$= \dfrac{r_1^2 - r r_2}{r_1^2(1+\tan^2\phi)}$

$\dfrac{d\phi}{d\theta} = \dfrac{r_1^2 - r r_2}{r_1^2\left[1+\frac{r^2}{r_1^2}\right]}$

$= \dfrac{r_1^2 - r r_2}{\cancel{r_1^2}\left[\frac{r_1^2+r^2}{\cancel{r_1^2}}\right]}$

$\dfrac{d\phi}{d\theta} = \dfrac{r_1^2 - r r_2}{r_1^2 + r^2}$

add 1 on b.s

$1+\dfrac{d\phi}{d\theta} = 1 + \dfrac{r_1^2 - r r_2}{r_1^2 + r^2}$

$= \dfrac{r_1^2 + r^2 + r_1^2 - r r_2}{r_1^2 + r^2}$

$\therefore 1+\dfrac{d\phi}{d\theta} = \dfrac{r^2 + 2r_1^2 - r r_2}{r_1^2 + r^2} \rightarrow (2)$

Also $\dfrac{ds}{d\theta} = \sqrt{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}$ [from the derivative of arclength]

$= \sqrt{r^2 + r_1^2} \rightarrow (3)$

Sub (2) & (3) in (1),

$$\rho = \frac{\sqrt{r^2+r_1^2} \times (r^2+r_1^2)}{r^2+2r_1^2-rr_2}$$

$$\boxed{\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}}$$

① Show that for the curve $r(1-\cos\theta) = 2a$, ρ^2 varies as r^3

$$r(1-\cos\theta) = 2a \rightarrow ①$$

$$\log[r(1-\cos\theta)] = \log 2a$$

$$\log r + \log(1-\cos\theta) = \log 2a$$

diff w.r.t θ.

$$\frac{1}{r}\cdot\frac{dr}{d\theta} + \frac{1}{1-\cos\theta} \times \sin\theta = 0$$

$$\frac{r_1}{r} = \frac{-\sin\theta}{1-\cos\theta}$$

$$\frac{r_1}{r} = \frac{-2\sin\theta/2 \cdot \cos\theta/2}{2\sin^2\theta/2}$$

$$\frac{r_1}{r} = -\cot\left(\frac{\theta}{2}\right)$$

$$\boxed{r_1 = -r\cot\theta/2}$$

diff w.r.t θ.

$$r_2 = -\left[r\left\{-\operatorname{cosec}^2(\theta/2)(1/2)\right\} + \cot\theta/2 \cdot r_1\right]$$

$$r_2 = -\left[-r/2\,\operatorname{cosec}^2(\theta/2) + r_1\cot(\theta/2)\right]$$

$$r_2 = \frac{r}{2}\operatorname{cosec}^2\left(\frac{\theta}{2}\right) - r_1\cot\left(\frac{\theta}{2}\right)$$

$$\boxed{r_2 = \frac{r}{2}\operatorname{cosec}^2\left(\frac{\theta}{2}\right) + r\cot^2\left(\frac{\theta}{2}\right)}$$ Sub for r_1

R.O.C, $$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

$$\rho = \frac{\left[r^2+r^2\cot^2(\theta/2)\right]^{3/2}}{r^2+2r^2\cot^2(\theta/2) - r\left\{r/2\,\operatorname{cosec}^2(\theta/2) + r\cot^2(\theta/2)\right\}}$$

$$\rho = \frac{(r^2)^{3/2}\left[1+\cot^2\theta/2\right]^{3/2}}{r^2+2r^2\cot^2(\theta/2) - r^2/2\,\operatorname{cosec}^2(\theta/2) - r^2\cot^2(\theta/2)}$$

$$\rho = \frac{r^3(\operatorname{cosec}^2 \theta/2)^{3/2}}{r^2 + r^2\cot^2(\theta/2) - r^2/2 \operatorname{cosec}^2(\theta/2)}$$

$$\rho = \frac{r^3 \operatorname{cosec}^3(\theta/2)}{r^2[1+\cot^2(\theta/2)] - r^2/2 \operatorname{cosec}^2(\theta/2)}$$

$$\rho = \frac{r^3 \operatorname{cosec}^3(\theta/2)}{r^2 \operatorname{cosec}^2(\theta/2) - r^2/2 \operatorname{cosec}^2(\theta/2)}$$

$$= \frac{r^3 \operatorname{cosec}^3(\theta/2)}{\frac{1}{2} r^2 \operatorname{cosec}^2(\theta/2)}$$

$$\rho = 2r \operatorname{cosec}(\theta/2) \rightarrow (2)$$

To eliminate θ.

from (1), $r(1-\cos\theta) = 2a$.

$r \times \cancel{2}\sin^2\left(\frac{\theta}{2}\right) = \cancel{2}a.$

$$\frac{r}{a} = \operatorname{cosec}^2\left(\frac{\theta}{2}\right)$$

$$\therefore \operatorname{cosec}\left(\frac{\theta}{2}\right) = \sqrt{\frac{r}{a}}.$$

Sub in (2).

$$\rho = 2r\sqrt{\frac{r}{a}}.$$

squaring,

$$\rho^2 = 4r^2 \times \frac{r}{a}$$

$$\rho^2 = \frac{4}{a} r^3$$

$$\rho^2 = (\text{const})\, r^3$$

$$\rho^2 \propto r^3$$

(5) Find the R.O.C of the polar curve $r^2 = a^2\cos 2\theta$

$$r^2 = a^2\cos 2\theta \rightarrow (1)$$

$$\log r^2 = \log[a^2\cos 2\theta]$$

$$2\log r = \log a^2 + \log\cos 2\theta$$

diff. w.r.to θ

$$\cancel{r}\ \frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{\cos 2\theta} \times -\sin 2\theta \times \cancel{2}$$

$$\frac{r_1}{r} = -\tan 2\theta$$

$$\boxed{r_1 = -r\tan 2\theta} \rightarrow (2)$$

diff w.r.t θ.

$$r_2 = -[r\sec^2 2\theta \times 2 + \tan 2\theta \times r_1]$$

$$r_2 = -2r\sec^2 2\theta - r_1\tan 2\theta.$$

$$\boxed{r_2 = -2r\sec^2 2\theta + r\tan^2 2\theta}. \text{ (using 2)}$$

R.O.C. $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1-rr_2}$

$$\rho = \frac{(r^2+r^2\tan^2 2\theta)^{3/2}}{r^2+2r^2\tan^2 2\theta - r\{-2r\sec^2 2\theta + r\tan^2 2\theta\}}$$

$$\rho = \frac{(r^2)^{3/2}\,[1+\tan^2 2\theta]^{3/2}}{r^2+2r^2\tan^2 2\theta + 2r^2\sec^2 2\theta - \underline{r^2\tan^2 2\theta}}$$

$$\rho = \frac{r^3\,[\sec^2 2\theta)^{3/2}}{r^2+r^2\tan^2 2\theta + 2r^2\sec^2 2\theta}$$

$$\rho = \frac{r^3\sec^3 2\theta}{r^2(1+\tan^2 2\theta) + 2r^2\sec^2 2\theta}.$$

$$\rho = \frac{r^3\sec^3 2\theta}{r^2\sec^2 2\theta + 2r^2\sec^2 2\theta}$$

$$\rho = \frac{r^3\sec^3 2\theta}{3r^2\sec^2 2\theta}$$

$$\rho = \frac{1}{3}r\sec 2\theta \rightarrow (3)$$

To eliminate θ

from (1), $r^2 = a^2\cos 2\theta$

$$\frac{1}{\cos 2\theta} = \frac{a^2}{r^2}$$

$$\sec 2\theta = \frac{a^2}{r^2} \rightarrow (4)$$

sub (4) in (3)

$$\rho = \frac{1}{3}\cdot r\cdot \frac{a^2}{r^2}$$

$$\boxed{\rho = \frac{a^2}{3r}}$$

$$\rho = \left(\frac{a^2}{3}\right)\frac{1}{r}.$$

$$\rho = (\text{const})\frac{1}{r}$$

$$\rho \propto \frac{1}{r}.$$

③ Find the R.O.C of the curve $r = a(1+\cos\theta)$

$$r = a(1+\cos\theta) \rightarrow ①$$

$$\log r = \log[a(1+\cos\theta)]$$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \log a + \log(1+\cos\theta)$$

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = 0 + \frac{1}{1+\cos\theta} \times -\sin\theta$$

$$\frac{r_1}{r} = \frac{-\sin\theta}{1+\cos\theta}$$

$$\frac{r_1}{r} = \frac{-\cancel{2}\sin\theta/_2\cdot\cos\theta/_2}{\cancel{2}\cos^2\theta/_2}$$

$$\frac{r_1}{r} = -\frac{\sin\theta/_2}{\cos\theta/_2}$$

$$\frac{r_1}{r} = -\tan\theta/_2 \qquad \boxed{r_1 = -r\tan\theta/_2} \rightarrow ②$$

diff w.r.t θ.

$$r_2 = -\left[r.\sec^2\theta/_2\cdot 1/_2 + \tan\theta/_2\, r_1\right]$$

$$r_2 = -r/_2\sec^2\theta/_2 - r_1\tan\theta/_2.$$

$$\boxed{r_2 = -r/_2\sec^2\theta/_2 + r\tan^2\theta/_2} \quad (\text{using } 2)$$

R.O.C $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1 - rr_2}$

$$\rho = \frac{(r^2+r^2\tan^2\theta/_2)^{3/2}}{r^2+2r^2\tan^2(\theta/_2) - r\{-r/_2\sec^2\theta/_2 + r\tan^2\theta/_2\}}$$

$$\rho = \frac{(r^2)^{3/2}(1+\tan^2\theta/_2)^{3/2}}{r^2+2r^2\tan^2(\theta/_2) + r^2/_2\sec^2\theta/_2 - r^2\tan^2\theta/_2}$$

$$\rho = \frac{r^3 (\sec^2 \theta/2)^{3/2}}{r^2 + r^2 \tan^2(\theta/2) + r^2/2 \sec^2(\theta/2)}$$

$$\rho = \frac{r^3 \sec^3 \theta/2}{r^2(1+\tan^2(\theta/2)) + r^2/2 \sec^2(\theta/2)}$$

$$\rho = \frac{r^3 \sec^3 \theta/2}{r^2 \sec^2 \theta/2 + r^2/2 \sec^2(\theta/2)}$$

$$\rho = \frac{r^3 \sec^3 \theta/2}{3/2\, r^2 \sec^2(\theta/2)}$$

$$\rho = \frac{2}{3} r \sec(\theta/2) \rightarrow (3)$$

To eliminate θ

from (1), $r = a(1+\cos\theta)$

$$r = a \cdot 2\cos^2\theta/2$$

$$\frac{1}{\cos^2\theta/2} = \frac{2a}{r}$$

$$\sec^2\theta/2 = \frac{2a}{r}$$

$$\sec\theta/2 = \sqrt{\frac{2a}{r}} \rightarrow (4)$$

Sub (4) in (3)

$$\rho = \frac{2}{3} r \cdot \sqrt{\frac{2a}{r}}$$

Squaring on b.s

$$\rho^2 = \frac{4}{9} r^2 \cdot \frac{2a}{r}.$$

$$= \frac{4}{9} r\, 2a.$$

$$\boxed{\rho^2 = \frac{8ar}{9}.}$$

(4) Find the R.O.C of the curve $r^n = a^n \sin n\theta$

$$r^n = a^n \sin n\theta \rightarrow (1)$$

$$\log r^n = \log a^n + \log(\sin n\theta)$$

$$n \log r = \log a^n + \log(\sin n\theta)$$

diff w.r.to θ

$$r \cdot \frac{1}{r} \cdot \frac{dr}{d\theta} = 0 + \frac{1}{\sin n\theta} \times \cos n\theta \times n$$

$$\frac{r_1}{r} = \frac{\cos n\theta}{\sin n\theta}$$

$$\boxed{r_1 = r \cot n\theta} \rightarrow (2)$$

diff w.r.t θ

$$r_2 = -r \cdot \operatorname{cosec}^2 n\theta \times n + \cot n\theta \times r_1$$

$$\boxed{r_2 = -rn \operatorname{cosec}^2(n\theta) + r \cdot \cot^2 n\theta} \quad (\text{using } (2))$$

R.O.C $\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}$

$$\rho = \frac{(r^2 + r^2 \cot^2 n\theta)^{3/2}}{r^2 + 2r^2 \cot^2 n\theta - r\{-rn \operatorname{cosec}^2(n\theta) + r \cdot \cot^2 n\theta\}}$$

$$\rho = \frac{(r^2)^{3/2} (1 + \cot^2 n\theta)^{3/2}}{r^2 + 2r^2 \cot^2 n\theta + r^2 n \operatorname{cosec}^2(n\theta) - r^2 \cot^2 n\theta}$$

$$\rho = \frac{r^3 (1 + \cot^2 n\theta)^{3/2}}{r^2 + r^2 \cot^2 n\theta + r^2 n \operatorname{cosec}^2(n\theta)}$$

$$\rho = \frac{r^3 (\operatorname{cosec}^2 n\theta)^{3/2}}{r^2 + r^2 \cot^2 n\theta + r^2 n \operatorname{cosec}^2(n\theta)}$$

$$= \frac{r^3 \operatorname{cosec}^3 n\theta}{r^2(1 + \cot^2 n\theta) + r^2 n \operatorname{cosec}^2 n\theta}$$

$$= \frac{r^3 \operatorname{cosec}^3 n\theta}{r^2 \operatorname{cosec}^2(n\theta) + r^2 n \operatorname{cosec}^2 n\theta}$$

$$= \frac{r^3 \operatorname{cosec}^3 n\theta}{(1+n) r^2 \operatorname{cosec}^2(n\theta)}$$

$$\boxed{\rho = \frac{1}{(n+1)} r \cdot \operatorname{cosec}(n\theta)} \rightarrow (3)$$

To eliminate θ

from (1), $r^n = a^n \sin n\theta$

$$\frac{1}{\sin n\theta} = \frac{a^n}{r^n}$$

$$\operatorname{cosec} n\theta = \frac{a^n}{r^n} \rightarrow (4)$$

Sub ④ in ③

$$\rho = \frac{1}{(n+1)} r \cdot \frac{a^n}{r^n}$$

$$\boxed{\rho = \frac{a^n}{(n+1) r^{n-1}}}$$

Radius of curvature in the Pedal Form.

$$\rho = r \frac{dr}{dp}.$$

① Show that the R.O.C of the curve $pa^2 = r^3$ is $\frac{a^2}{3r}$

$$pa^2 = r^3$$

diff w.r.t p.

$$a^2 \cdot 1 = 3r^2 \frac{dr}{dp}$$

$$\frac{dr}{dp} = \frac{a^2}{3r^2}$$

$$\text{R.O.C } \rho = r \cdot \frac{dr}{dp}$$

$$= r \cdot \frac{a^2}{3r^2}$$

$$\boxed{\rho = \frac{a^2}{3r}.}$$

② S.T for the curve $p^2 = ar$. ρ^2 varies as r^3

$$p^2 = ar.$$

diff w.r.t p.

$$2p = a \cdot \frac{dr}{dp}$$

$$\frac{dr}{dp} = \frac{2p}{a}$$

R.O.C $\rho = r\dfrac{dr}{dp}$

$= r \cdot \dfrac{2p}{a}$.

To eliminate p.

$\rho = \dfrac{2rp}{a}$

$= \dfrac{2r}{a}\sqrt{ar}$.

$\rho^2 = \dfrac{4r^2}{a^2}\, ar$.

$\rho^2 = \dfrac{4r^3}{a}$

$\rho^2 \propto r^3$

③ Given $p^2 = \dfrac{r^3}{2a}$ S.T $\dfrac{\rho^2}{r}$ is const.

$p^2 = \dfrac{r^3}{2a}$

diff w.r.t p.

$2ap^2 = r^3$

$2a\, 2p = 3r^2 \dfrac{dr}{dp}$.

$\dfrac{dr}{dp} = \dfrac{4ap}{3r^2}$

R.O.C $\rho = r\dfrac{dr}{dp}$

$= r \cdot \dfrac{4ap}{3r^2}$

$= \dfrac{4ap}{3r}$

$p^2 = \dfrac{r^3}{2a} \Rightarrow \sqrt{\dfrac{r^3}{2a}}$ (To eliminate p from ①)

$$\rho = \frac{4a}{3r}\sqrt{\frac{r^3}{2a}}$$

$$\rho^2 = \frac{16a^2}{9r^2} \cdot \frac{r^3}{2a}$$

$$\rho^2 = \left(\frac{8a}{9}\right) r$$

$$\frac{\rho^2}{r} = \text{const.}$$

④ Given $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$, S.T ROC is $\frac{a^2b^2}{p^3}$

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$$

diff w.r.t p.

$$\frac{-2}{p^3} = 0 + 0 - \frac{1}{a^2b^2} \cdot 2r\frac{dr}{dp}$$

$$\frac{\cancel{-2}}{p^3} = \cancel{-}\frac{\cancel{2}r}{a^2b^2} \cdot \frac{dr}{dp}$$

$$\frac{dr}{dp} = \frac{a^2b^2}{p^3 r}$$

$$\text{R.O.C } \rho = r\frac{dr}{dp}$$

$$= \cancel{r} \cdot \frac{a^2b^2}{p^3\cancel{r}}$$

$$\rho = \frac{a^2b^2}{p^3}$$

⑤ $r\cos^2(\theta/2) = a$.

$$\log r + \log\cos^2\theta/2 = 0$$

$$\frac{1}{r}\frac{dr}{d\theta} \cdot \frac{2\cos\theta/2}{\cos^2\theta/2} \cdot \sin\theta/2\,(1/2)$$

$$r_1 = -r\tan\theta/2$$

$$r_1 = r\tan\theta/2$$

diff w.r.t θ

$$r_2 = r\left(\sec^2 \theta/2 \, (1/2) + \tan \theta/2 \; r_1\right)$$
$$= \frac{r}{2}\left(\sec^2 \theta/2 + r \tan^2 \theta/2\right)$$

$$\rho = \left[\frac{\left(r^2 + (r_1)^2\right)^{3/2}}{r^2 + 2r_1^2 - r r_2}\right]$$

$$= \frac{r^2 + \left(r^2 + \tan^2 \theta/2\right)^{3/2}}{r^2 + 2r^2 \tan^2 \theta/2 - r\left[r/2 \sec^2 \theta/2 + r \tan^2 \theta/2\right]}$$

$$= \frac{(r^2)^{3/2} \left(1 + \tan^2 \theta/2\right)^{3/2}}{r^2 + 2r^2 \tan^2 \theta/2 - r^2/2 \sec^2 \theta/2 + r^2 \tan^2 \theta/2}$$

$$= \frac{r^3 \left(\sec^3 \theta/2\right)}{r^2 + r^2 \tan^2 \theta/2 \left[r^2/2 \sec^2 \theta/2\right]}$$

$$= \frac{r^3 \sec^3 \theta/2}{(1/2) r^2 \left(1 + \tan^2 \theta/2\right)\left(\sec^2 \theta/2\right)}$$

$$= \frac{r^3 \sec^3 \theta/2}{(1 - 1/2) r^2 \left(\sec^2 \theta/2\right)}$$

$$= \frac{r^3 \sec^3 \theta/2}{2 r^2 \sec^2 \theta/2}$$

$$\rho = 2r \sec \theta/2$$

To eliminate θ

$$r \cos^2(\theta/2) = a$$

$$\frac{a}{r} = \frac{1}{\cos^2 \theta/2} = \sec^2 \theta/2 = \frac{a}{r} = \sqrt{\frac{a}{r}}$$

$$= 2r\sqrt{\frac{a}{r}}$$

$$= 2^2 r^2 \, a/r$$

$$\rho^2 = 4a/r^3$$

$$\rho \propto r^3$$

(6) $2a/r = (1 + \cos\theta)$

$$\log 2a + \log r = \frac{1}{1 + \cos\theta} (\sin\theta)$$

$$r/r_1 = \frac{\sin\theta}{(1 + \cos\theta)}$$

$$\frac{r}{r_1} = \frac{2 \sin \theta/2 \cdot \cos \theta/2}{2 \cos^2 \theta/2}$$

$r_1 = r\tan\theta/2$

diff w.r.t θ

$r_2 = r/2\left(\sec^2\theta/2\,(1/2)\, r\tan^2\theta/2\right)$

$$= \frac{\left[(r^2+(r_1)^2)^{3/2}\right]}{r^2+2r_1^2-r r_2}$$

$$= \frac{\left[r^2+(r^2\tan^2\theta/2)\right]^{3/2}}{r^2+2r^2\tan^2\theta/2 - r\left[r/2\,\sec^2\theta/2 + r\tan^2\theta/2\right]}$$

$$= \frac{[r^2]^{3/2}\left[1+\tan^2\theta/2\right]^{3/2}}{r^2+2r^2\tan^2\theta/2 - r^2/2\,\sec^2\theta/2 + r^2\tan^2\theta/2}$$

$$= \frac{r^3\sec^3\theta/2}{r^2+r^2\tan^2\theta/2 - r^2/2\,\sec^2\theta/2}$$

$$= \frac{r^3\sec^3\theta/2}{(1-1/2)(r^2\sec^2\theta/2)}$$

$$= \frac{r\sec\theta/2}{2}$$

To eliminate θ

$$\frac{2a}{r} = (1+\cos\theta)$$

$$\frac{2a}{r} = 2\cos^2\theta/2$$

$$\frac{r}{2a} = 2\sec^2\theta/2$$

$$\sec\theta/2 = \sqrt{r/a}.$$

$$= 2r\left[\frac{\sqrt{r}}{a}\right]$$

$$= 4r^2\left[\frac{r}{a}\right]$$

$$\rho = \frac{4r^3}{a}$$